On the properties of well-graded partially union-closed families

Jeffrey Matayoshi

McGraw-Hill Education/ALEKS Corporation, 15460 Laguna Canyon Road, Irvine, CA, 92618, United States

Abstract

In this paper we will study several properties of well-graded union-closed families that do not contain the empty set. Such union-closed families without the empty set are said to be partially union-closed. We will extend several results for well-graded union-closed families to the partially union-closed case, and we will also extend the concept of being intersection-closed to families without the empty set.

Keywords: Knowledge spaces, Learning spaces, Well-graded, Partially union-closed, Upper intersection-closed
2010 MSC: 06A07, 91E45

1. Introduction

Families of sets that are $\cup$-closed are of interest for both their theoretical properties and their use in practical applications. In particular, knowledge spaces are $\cup$-closed families of sets that have found many successful uses in the assessment of knowledge (Doignon and Falmagne, 1985; Falmagne and Doignon, 2011; Falmagne et al., 2013).

Definition 1.1. A *knowledge structure* is a pair $(Q, \mathcal{K})$ in which Q is a nonempty set, and $\mathcal{K}$ is a family of subsets of Q, containing at least Q and the empty set $\varnothing$. The set Q is called the *domain* of the knowledge structure. Its elements are referred to as *questions* or *items* and the subsets in the family $\mathcal{K}$ are labeled *(knowledge) states.* Since $\cup\mathcal{K} = Q$, we shall sometimes simply say that $\mathcal{K}$ is the knowledge structure when reference to the underlying domain is not necessary. If a knowledge structure $\mathcal{K}$ is closed under union, we say that $\mathcal{K}$ is a *knowledge space.*

A useful concept associated with $\cup$-closed families is well-gradedness, which we will define as in Doignon and Falmagne (1997).

Email address: `jeffrey.matayoshi@aleks.com` (Jeffrey Matayoshi)
[1]To appear in the Journal of Mathematical Psychology

Definition 1.2. Let Δ denote the standard symmetric difference between sets. Then, a family of sets $\mathcal{F}$ is *well-graded* if for any $A, B \in \mathcal{F}$ with $|A\Delta B| = n$, there exists a finite sequence of sets $A = K_0, K_1, \ldots, K_n = B$ in $\mathcal{F}$ such that $|K_{i-1}\Delta K_i| = 1, i = 1, \ldots, n$. The sequence of sets $A = K_0, K_1, \ldots, K_n = B$ satisfying these conditions is called a *tight path* between A and B.

If a knowledge space is well-graded, Cosyn and Uzun (2009) showed that we have a learning space, a special type of knowledge space whose properties are motivated by pedagogical assumptions. One subtle assumption is that the empty set $\varnothing$ is necessarily part of a learning space. While this may not seem like an important assumption at first glance, in what follows we will see that many of the properties of learning spaces, as well as the techniques used to study these properties, depend heavily on the inclusion of the empty set; thus, extra complications arise when the inclusion of the empty set is not guaranteed.

In this paper we will focus on the properties of well-graded $\cup$-closed families of sets that do not contain the empty set; such families are said to be well-graded and partially $\cup$-closed. As mentioned in the previous paragraph, the lack of an empty set presents obstacles that will require different techniques to handle compared to those used for a normal $\cup$-closed family. Starting with several properties of learning spaces, we will derive analogous results for well-graded partially $\cup$-closed families of sets. Furthermore, we will also extend the concept of an ordinal space (i.e., a discriminative learning space that is also $\cap$-closed) to families of sets that do not contain the empty set. Then, after proving a result for projections of such families, we will finish by looking at the infinite case. Along the way we will have provided possible solutions to two of the open problems mentioned in Section 18.2 of Falmagne and Doignon (2011).

In addition to the interesting theoretical challenges that result from the lack of an empty set, there are practical reasons for studying such families. As described in Falmagne (2008) and Chapters 2 and 13 of Falmagne and Doignon (2011), partially $\cup$-closed families may be encountered when using projections of knowledge spaces. Such projections of knowledge spaces have found applications in assessments of knowledge where, in many cases, it may be unwieldy, or even impossible, to run an assessment over a full knowledge space.

As another example of a practical application, when a $\cup$-closed family is being used to represent a domain of knowledge, one can make the argument that the empty set is not a realistic state in many situations. In an implementation of knowledge spaces such as the artificial intelligence used in the ALEKS system, having a student in a knowledge space with the empty set as their state would seem to indicate that the student is misplaced; in reality, all students have some level of knowledge, so it is likely that there exists a different knowledge space that would be a better fit for such a student. Under this viewpoint, a student who is placed in a properly designed domain of knowledge should never start in the empty state. The benefit is that the family of sets can then be engineered and built without having to necessarily include the empty set. Thus, by starting from a collection of minimal nonempty sets, the entire family can be built without needing to worry about the empty set or any other sets contained in these minimal states. In essence, the process can be simplified by not having to worry about the "bottom" of the family of sets.

2. Background

Motivated by pedagogical assumptions, Cosyn and Uzun (2009) introduced two axioms that define a learning space (note that, with the exception of Section 6, we will assume throughout this paper that we are dealing with a finite family of sets on a finite domain of items).

Definition 2.1. A knowledge structure $(Q, \mathcal{K})$ is called a learning space if it satisfies the following conditions.

[L1] LEARNING SMOOTHNESS. For any two states K, L such that $K \subset L$, there exists a finite chain of states

$$K = K_0 \subset K_1 \subset \cdots \subset K_p = L$$

such that $|K_i \setminus K_{i-1}| = 1$ for $1 \leq i \leq p$ and so $|L \setminus K| = p$.

[L2] LEARNING CONSISTENCY. If K, L are two states satisfying $K \subset L$ and q is an item such that $K \cup \{q\} \in \mathcal{K}$, then $L \cup \{q\} \in \mathcal{K}$.

Cosyn and Uzun showed that a learning space, characterized by these axioms, is equivalent to a well-graded $\cup$-closed family.

Theorem 2.2 (Cosyn and Uzun)**.** *Let $\mathcal{F}$ be a family of sets containing the empty set. Then, $\mathcal{F}$ is well-graded and $\cup$-closed if and only if [L1] and [L2] are satisfied. In other words, well-graded $\cup$-closed families of sets are characterized by axioms [L1] and [L2].*

The example below (copied from Example 2.2.8 in Falmagne and Doignon, 2011) shows that Theorem 2.2 fails to hold when $\mathcal{F}$ does not contain the empty set. In particular, for a family $\mathcal{F}$ without the empty set, [L1] and [L2] do not guarantee that $\mathcal{F}$ is well-graded or $\cup$-closed.

Example 2.3. The family of sets

$$\mathcal{L} = \{\{a\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$$

satisfies [L1] and [L2]. However, $\mathcal{L}$ is neither $\cup$-closed nor well-graded.

In Section 3 we will derive a set of axioms that gives a result analogous to Theorem 2.2 when $\mathcal{F}$ does not contain the empty set. To do this, we will make use of the terminology in the following definition from Falmagne and Doignon (2011).

Definition 2.4. A family $\mathcal{F}$ of subsets of a nonempty set Q is a *partial knowledge structure* if it contains the set $Q = \cup\mathcal{F}$. We also call the sets in $\mathcal{F}$ *states*. A partial knowledge structure $\mathcal{F}$ is a *partial learning space* if it satisfies axioms [L1] and [L2]. A family $\mathcal{F}$ is *partially $\cup$-closed* if for any nonempty subfamily $\mathcal{G}$ of $\mathcal{F}$, we have $\cup\mathcal{G} \in \mathcal{F}$. (Contrary to the $\cup$-closure condition, partial $\cup$-closure does not imply that the empty set belongs to the family). A *partial knowledge space* $\mathcal{F}$ is a partial knowledge structure that is partially $\cup$-closed.

We will also need the following definition from Eppstein et al. (2009).

Definition 2.5. Let $\mathcal{F}$ be a nonempty family of sets. For any $q \in Q = \cup\mathcal{F}$, an *atom at* q is a minimal set of $\mathcal{F}$ containing q (where 'minimal' is with respect to set inclusion). A set $X \in \mathcal{F}$ is called an *atom* if it is an atom at q for some $q \in Q$. We denote by $\sigma(q)$ the collection of all atoms at q, and we call $\sigma : Q \to 2^{2^Q}$ the *surmise function* of $\mathcal{F}$. We say that σ is *discriminative* if whenever $\sigma(q) = \sigma(q')$ for some $q, q' \in Q$, then $q = q'$. In such a case, we also refer to the family $\mathcal{F}$ as *discriminative*.

Throughout this paper we will assume that $Q = \cup\mathcal{F}$ and, with the exception of Section 6, that Q is a finite set.

3. Axioms for well-graded partially ∪-closed families

Consider the following axioms for a family $\mathcal{F}$ of sets (states) that does not contain the empty set.

Definition 3.1.
[L1] (same as in Definition 2.1) For any two states K, L such that $K \subset L$, there exists a finite chain of states

$$K = K_0 \subset K_1 \subset \cdots \subset K_p = L$$

such that $|K_i \setminus K_{i-1}| = 1$ for $1 \leq i \leq p$ and so $|L \setminus K| = p$.

[L2*] Let A be an atom, and let $q_1, \ldots, q_n \in Q$ be all the items in Q at which A is an atom. If L is a state such that $A \setminus \{q_1, \ldots, q_n\} \subseteq L$, then $L \cup \{q_1, \ldots, q_n\} \in \mathcal{F}$.

The next result shows that, in the case of a family of sets containing $\varnothing$, [L1] and [L2*] are equivalent to [L1] and [L2].

Theorem 3.2. *Let $\mathcal{F}$ be a family of sets. Then, [L1] and [L2] hold whenever [L1] and [L2*] hold. Furthermore, [L1] and [L2] are equivalent to [L1] and [L2*] when $\mathcal{F}$ contains the empty set.*

Proof.
We will first show that [L1] and [L2] are implied by [L1] and [L2*]. Note that for this initial result we will not assume that $\mathcal{F}$ contains the empty set. Let $K, L \in \mathcal{F}$ with $K \subset L$, and let $q \in Q$ be such that $K \cup \{q\} \in \mathcal{F}$. For [L2] to hold, we need to show $L \cup \{q\} \in \mathcal{F}$. Let A be an atom at q such that $A \subseteq K \cup \{q\}$. For $q_1 = q$ and $n \geq 1$, let $\{q_1, \ldots, q_n\} \subseteq A$ be the set composed of all the items at which A is an atom. It follows that $A \setminus \{q_1, \ldots, q_n\} \subseteq K \subset L$, and by [L2*] we have $L \cup \{q\} = L \cup \{q_1, \ldots, q_n\} \in \mathcal{F}$.

We will next assume that $\mathcal{F}$ contains the empty set. Given [L1] and [L2], it is shown in Cosyn and Uzun (2009) that $\mathcal{F}$ is a well-graded ∪-closed family. By Theorem 5.4.1 in Falmagne and Doignon (2011) (which is a generalization of a result from Koppen, 1998) any atom in $\mathcal{F}$ is an atom at exactly one item.

Letting $A, L \in \mathcal{F}$, where A is an atom at q and $A \setminus \{q\} \subseteq L$, by the remark in the previous sentence we know that q is the only item at which A is an atom. Thus, [L2*] will hold if we can show that $L \cup \{q\} \in \mathcal{F}$. This follows directly from the $\cup$-closure property of $\mathcal{F}$ since $L \cup \{q\} = L \cup A \in \mathcal{F}$. □

While [L2*] follows the spirit of [L2], the next result shows that [L2*] is actually a much stronger condition.

Theorem 3.3. *Let $\mathcal{F}$ be a family of sets. Then $\mathcal{F}$ is partially $\cup$-closed if and only if [L2*] holds.*[2]

Proof.
$\mathcal{F}$ partially $\cup$-closed $\Rightarrow$ [L2*]:

Let A be an atom, and let $q_1, \ldots, q_n \in Q$ be all the items in Q at which A is an atom. Let L be a state such that $A \setminus \{q_1, \ldots, q_n\} \subseteq L$. To show [L2*] holds, we need to show that $L \cup \{q_1, \ldots, q_n\}$ is in $\mathcal{F}$. Since both L and A are states in $\mathcal{F}$, by hypothesis $L \cup A \in \mathcal{F}$. Thus, we have $L \cup \{q_1, \ldots, q_n\} = L \cup A \in \mathcal{F}$.

[L2*] $\Rightarrow$ $\mathcal{F}$ partially $\cup$-closed:

Let $K, L \in \mathcal{F}$. Note that without loss of generality we can assume K and L are not subsets of each other; otherwise, it would follow trivially that $K \cup L \in \mathcal{F}$. We will start by claiming that we can always find a nonempty set $D \subseteq K \setminus L$ such that $D \cup L \in \mathcal{F}$. Under this assumption, we can then find another nonempty set $D' \subseteq K \setminus (D \cup L)$ such that $D' \cup D \cup L \in \mathcal{F}$. Iteratively repeating this procedure, by the finiteness of K and L we will eventually conclude that $K \cup L \in \mathcal{F}$, as desired.

To that end, let q_0 be in $K \setminus L$, and let A_0 be an atom at q_0 such that $A_0 \subseteq K$. Let C_0 be the set of all items at which A_0 is an atom; thus, $q_0 \in C_0$. Now, if $A_0 \setminus C_0 \subseteq L$, it follows by [L2*] that $C_0 \cup L \in \mathcal{F}$. Otherwise, from the remaining items in $K \setminus (C_0 \cup L)$ we can choose another item, q_1. Let $A_1 \subseteq K$ be an atom at q_1, and let C_1 be the set of all items at which A_1 is an atom. If $A_1 \setminus C_1 \subseteq L$, it follows by [L2*] that $C_1 \cup L \in \mathcal{F}$. If not, we can continue the same procedure. Thus, after n iterations we have q_n, A_n and C_n, where $q_n \in K \setminus \left(\bigcup_{i=0}^{n-1} C_i \cup L\right)$, $A_n \subseteq K$ is an atom at q_n, and C_n contains all the items at which A_n is an atom. By the finiteness of our space, it must be true that eventually $A_n \setminus C_n \subseteq L$ for some n; by [L2*] it follows that $C_n \cup L \in \mathcal{F}$.

Once we have C_n, we can define $D := C_n \cap (K \setminus L)$. Since $q_n \in K \setminus \left(\bigcup_{i=0}^{n-1} C_i \cup L\right) \subseteq K \setminus L$, we know that $D \neq \varnothing$. Furthermore, we have $D \cup L = C_n \cup L \in \mathcal{F}$, and the claimed result follows. □

[2]The statement of Theorem 3.3, along with a detailed outline of the proof, were generously communicated to the author by Jean-Paul Doignon in his review of this paper. The implications of Theorem 3.3, in addition to being of interest on their own, are notable for significantly shortening the proof of Corollary 3.4.

The next result, which follows from Theorem 3.3, will show that axioms [L1] and [L2*] completely characterize well-graded partially $\cup$-closed families; thus, it is a possible solution to the following open problem.

Strengthening [L1] and [L2] (Problem 18.2.1 from Falmagne and Doignon, 2011)**.** *Any well-graded partially $\cup$-closed family is a partial learning space, but the converse implication does not hold. Find axioms that strengthen (or at least are in the spirit of) Axioms [L1] and [L2] that characterize well-graded partially $\cup$-closed families.*

Corollary 3.4. *Let $\mathcal{F}$ be a family of sets that does not contain the empty set. Then, $\mathcal{F}$ is well-graded and partially $\cup$-closed if and only if [L1] and [L2*] are satisfied.*

Proof.
$\mathcal{F}$ well-graded and partially $\cup$-closed $\Rightarrow$ [L1] and [L2*]:

For the necessity, [L1] is a consequence of well-gradedness, while [L2*] follows from partial $\cup$-closure and the fact that $L \cup \{q_1, \ldots, q_n\} = L \cup A$.

[L1] and [L2*] $\Rightarrow$ $\mathcal{F}$ well-graded and partially $\cup$-closed:

For the sufficiency, the fact that $\mathcal{F}$ is partially $\cup$-closed follows from [L2*] and Theorem 3.3. Once we have shown that $\mathcal{F}$ is partially $\cup$-closed, for arbitrary $K, L \in \mathcal{F}$ we can apply [L1] to K and $K \cup L$, and again to L and $K \cup L$, to get well-gradedness. □

4. The base of well-graded partially ∪-closed families

Definition 4.1. The *span* of a family of sets $\mathcal{G}$ is the family containing any set which is the union of any nonempty subfamily of $\mathcal{G}$. It follows that the empty set is in the span of $\mathcal{G}$ if and only if it is in $\mathcal{G}$ itself. The *base* of a (partially) $\cup$-closed family $\mathcal{F}$ is a minimal subfamily $\mathcal{B}$ of $\mathcal{F}$ spanning $\mathcal{F}$.

For a finite (partially) $\cup$-closed family $\mathcal{F}$, the base always exists and is unique. Furthermore, the base is composed of all the atoms in $\mathcal{F}$ (see Section 3.4 in Falmagne and Doignon, 2011, for more details). The following theorem from Eppstein et al. (2009) gives conditions for characterizing the base of a well-graded family.

Theorem 4.2 (Eppstein, Falmagne, and Uzun)**.** *Let $\mathcal{F}$ be a partially $\cup$-closed family with base $\mathcal{B}$. Then $\mathcal{F}$ is a well-graded family if and only if, for any two distinct sets K and L in $\mathcal{B}$, there is a tight path in $\mathcal{F}$ from K to $L \cup K$. If $\mathcal{B}$ contains the empty set, then $\mathcal{F}$ is well-graded if and only if there is a tight path from $\varnothing$ to K for any K in $\mathcal{B}$.*

The statement of Theorem 4.2 relies on properties of the family spanned by the base; however, as noted in the following open problem from Falmagne and Doignon (2011), it would be preferable if the statement relied only on the base itself.

Characterize well-graded spans (Problem 18.2.4 from Falmagne and Doignon, 2011). *Theorem 4.2 characterizes those families whose span is well-graded. However the characterization refers explicitly to the span. Find a characterization solely in terms of the spanning family.*

A possible solution is given by Theorem 4.7 below, the statement of which relies only on the base itself; the key idea for this theorem is given by the following result from Koppen (1998).

Theorem 4.3 (Koppen). *Let $\mathcal{K}$ be a knowledge space with surmise function σ and base $\mathcal{B}$. Then the following conditions are equivalent:*

(i) $\mathcal{K}$ is well-graded;

(ii) the family $\{\sigma(x) \mid x \in \cup\mathcal{K}\}$ is a partition of $\mathcal{B}$;

(iii) for any atom B at any item q, the set $B \setminus \{q\} \in \mathcal{K}$.

As the following example from Eppstein et al. (2009) shows, the above result fails to hold when we assume only that $\mathcal{K}$ is a partial knowledge space.

Example 4.4. Consider the partially $\cup$-closed family $\mathcal{F}$ with base

$$\mathcal{B} = \{\{x, y, c\}, \{y, d\}, \{c, d\}\}.$$

The surmise function σ is given by

$$\sigma(x) = \{\{x, y, c\}\}, \qquad \sigma(y) = \{\{x, y, c\}, \{y, d\}\},$$
$$\sigma(c) = \{\{x, y, c\}, \{c, d\}\}, \qquad \sigma(d) = \{\{y, d\}, \{c, d\}\}.$$

The family $\mathcal{F}$ is well-graded and σ is discriminative. However, each element of $\mathcal{B}$ is an atom at multiple items, so σ does not give a partition of the base; thus, (ii) fails. Furthermore, all of the atoms are minimal sets in the family spanned by $\mathcal{B}$, so (iii) fails as well. As an illustration of this last point, we can take the atom $\{y, d\}$ as an example and clearly see that neither $\{y, d\} \setminus \{y\} = \{d\}$ nor $\{y, d\} \setminus \{d\} = \{y\}$ are states.

The rest of this section will derive a set of theorems specific to well-graded partially $\cup$-closed families. In particular, we will present a version of Theorem 4.3 for partially $\cup$-closed families, and we will also discuss an alternative to Theorem 4.2 that relies only on the characteristics of the base. Finally, at the end of this section we will extend the concept of a "partial" space to the operation of intersections.

In what follows, we will need to make use of the following definition and lemma from Eppstein et al. (2009).

Definition 4.5. For any family of sets $\mathcal{G}$ and any set $X \in \mathcal{G}$, let $\mathcal{G} \setminus X$ denote the family of sets $\{Y \setminus X \mid Y \in \mathcal{G}\}$.

Lemma 4.6 (Eppstein, Falmagne, and Uzun)**.** *Let $\mathcal{B}$ be the base of a partially $\cup$-closed family $\mathcal{F}$. Then $\mathcal{F}$ is well-graded if and only if, for each X in $\mathcal{B}$, the family $\mathcal{B} \setminus X$ spans a learning space.*

Using Lemma 4.6 in combination with Theorem 4.3 gives the following result.

Theorem 4.7. *Let $\mathcal{F}$ be a partially $\cup$-closed family with base $\mathcal{B}$. Let $\sigma_{\mathcal{B}\setminus X}$ be the surmise function of the space spanned by $\mathcal{B} \setminus X$. Then the following conditions are equivalent:*

(i) $\mathcal{F}$ is well-graded;

(ii) for any $X \in \mathcal{B}$, the family $\{\sigma_{\mathcal{B}\setminus X}(x) \,|\, x \in \cup\mathcal{B} \setminus X\}$ is a partition of $\mathcal{B} \setminus X$;

(iii) for any $X \in \mathcal{B}$ the following holds: for any atom A at $q \in \cup\mathcal{B} \setminus X$ in the space spanned by $\mathcal{B} \setminus X$, the set $A \setminus \{q\}$ is a state.

In the rest of this section we will extend the concept of a partially closed family to intersections. We will start by proving the following lemma, which in its original form is due to Koppen (1998).

Lemma 4.8 (Koppen)**.** *A knowledge space $(Q, \mathcal{K})$ is $\cap$-closed if and only if each item $q \in Q$ has a unique atom.*

Proof.
$(Q, \mathcal{K})$ is $\cap$-closed $\Rightarrow$ each $q \in Q$ has a unique atom:

Suppose not. Then, there exists a $q \in Q$ such that $A, B \in \mathcal{K}$, $A \neq B$ are both atoms at q. However, $q \in A \cap B \in \mathcal{K}$, contradicting the assumption that A and B are atoms at q.

Each $q \in Q$ has a unique atom $\Rightarrow$ $(Q, \mathcal{K})$ is $\cap$-closed:

Let $K, L \in \mathcal{K}$. For each $q \in K \cap L$, let A_q be the unique atom at q. Note that $A_q \subseteq K \cap L$ since $A_q \subseteq K$ and $A_q \subseteq L$. It follows that $K \cap L = \left(\bigcup_{q \in K\cap L} A_q\right) \in \mathcal{K}$. □

Building on the concept of a partially $\cup$-closed family, we can define an analogous property for intersections.

Definition 4.9. A family of sets $\mathcal{F}$ is X*-closed* if for any nonempty subfamily $\mathcal{G}$ of $\mathcal{F}$, we have $\cap\mathcal{G} \in \mathcal{F}$ whenever $X \subseteq \cap\mathcal{G}$. When $\mathcal{F}$ is partially $\cup$-closed with base $\mathcal{B}$, we say that $\mathcal{F}$ is *upper* $\cap$*-closed* if $\mathcal{F}$ is X-closed for every $X \in \mathcal{B}$. In other words, any intersection of states that includes an element of the base is contained in $\mathcal{F}$.

The proof of our next theorem will make use of the following lemma.

Lemma 4.10. *Let $\mathcal{F}$ be a partially $\cup$-closed family with base $\mathcal{B}$. For any $X \in \mathcal{B}$, let $\mathcal{F}_{\mathcal{B}\setminus X}$ be the space spanned by $\mathcal{B} \setminus X$ and let $K \subseteq \cup\mathcal{B} \setminus X$. Then, $K \in \mathcal{F}_{\mathcal{B}\setminus X}$ if and only if there exists $\widetilde{K} \in \mathcal{F}$ such that $K = \widetilde{K} \setminus X$.*

Proof. We have

$$\begin{aligned} K \in \mathcal{F}_{\mathcal{B}\setminus X} &\iff K = \bigcup_{i=1}^{n} B_i \setminus X, \quad B_i \in \mathcal{B} \\ &\iff K = \left(\bigcup_{i=1}^{n} B_i\right) \setminus X \\ &\iff K = \widetilde{K} \setminus X, \end{aligned}$$

where $\widetilde{K} = (\bigcup_{i=1}^{n} B_i) \in \mathcal{F}$. □

Theorem 4.11. *Let $\mathcal{F}$ be a partially $\cup$-closed family with base $\mathcal{B}$, and fix any $X \in \mathcal{B}$. Then $\mathcal{F}$ is X-closed if and only if the space spanned by $\mathcal{B} \setminus X$ is $\cap$-closed. If $\mathcal{F}$ is discriminative, this is equivalent to saying that $\mathcal{F}$ is X-closed if and only if the space spanned by $\mathcal{B} \setminus X$ is an ordinal knowledge space (i.e., it is discriminative and both $\cup$-closed and $\cap$-closed).*

Proof.
$\mathcal{F}$ is X-closed $\Rightarrow$ $\mathcal{B} \setminus X$ spans an $\cap$-closed space:

Let $X \in \mathcal{B}$ and let $\mathcal{F}_{\mathcal{B}\setminus X}$ be the space spanned by $\mathcal{B} \setminus X$. Note that $\mathcal{F}_{\mathcal{B}\setminus X}$ is a knowledge space since $\varnothing = X \setminus X \in \mathcal{F}_{\mathcal{B}\setminus X}$. To show it is $\cap$-closed, let $K, L \in \mathcal{F}_{\mathcal{B}\setminus X}$. Then, by Lemma 4.10 there exist $\widetilde{K}, \widetilde{L} \in \mathcal{F}$ such that $K = \widetilde{K} \setminus X$ and $L = \widetilde{L} \setminus X$. Let $\widetilde{M} = (\widetilde{K} \cup X) \cap (\widetilde{L} \cup X)$. Then, $\widetilde{M} \in \mathcal{F}$ by the X-closure property since $X \subseteq \widetilde{M}$. We have

$$\begin{aligned} \widetilde{M} \setminus X &= ((\widetilde{K} \cup X) \cap (\widetilde{L} \cup X)) \setminus X \\ &= ((\widetilde{K} \cup X) \setminus X) \cap ((\widetilde{L} \cup X) \setminus X) \\ &= (\widetilde{K} \setminus X) \cap (\widetilde{L} \setminus X) \\ &= K \cap L. \end{aligned}$$

Applying Lemma 4.10 once more, it follows that $K \cap L \in \mathcal{F}_{\mathcal{B}\setminus X}$.

$\mathcal{B} \setminus X$ spans an $\cap$-closed space $\Rightarrow$ $\mathcal{F}$ is X-closed:

Let $\widetilde{K}, \widetilde{L} \in \mathcal{F}$ such that $X \subseteq \widetilde{K} \cap \widetilde{L}$. By Lemma 4.10 we have $K = \widetilde{K} \setminus X \in \mathcal{F}_{\mathcal{B}\setminus X}$ and $L = \widetilde{L} \setminus X \in \mathcal{F}_{\mathcal{B}\setminus X}$; since $\mathcal{B} \setminus X$ spans an $\cap$-closed space, we get that $K \cap L \in \mathcal{F}_{\mathcal{B}\setminus X}$. Thus, each $q \in K \cap L$ has an atom A_q in $\mathcal{F}_{\mathcal{B}\setminus X}$ such that $A_q \subseteq K \cap L$. It then follows that $K \cap L = \bigcup_{q \in K \cap L} A_q$. For each A_q, there exists $\widetilde{A}_q \in \mathcal{F}$ such that $A_q = \widetilde{A}_q \setminus X$. We then have $\widetilde{K} \cap \widetilde{L} = \bigcup_{q \in K \cap L} \widetilde{A}_q \cup X$. Note that the latter, being the union of sets in $\mathcal{F}$, is itself a member of $\mathcal{F}$. Thus, $\widetilde{K} \cap \widetilde{L} \in \mathcal{F}$, and it follows that $\mathcal{F}$ is X-closed.

To finish the proof of the second statement, we simply need to show that $\mathcal{F}_{\mathcal{B}\setminus X}$ is discriminative when $\mathcal{F}$ is discriminative. To see this, take any two items $q, r \in \cup\mathcal{B} \setminus X$. Assuming $\mathcal{F}$ is discriminative, there exists a state $M \in \mathcal{F}$ that contains only one of q or r (otherwise, q and r would share the same set of atoms, contradicting the assumption that $\mathcal{F}$ is discriminative). Without loss of generality, assume $q \in M$. Applying Lemma 4.10 one final time, we get that $M \setminus X \in \mathcal{F}_{\mathcal{B}\setminus X}$. Thus, $M \setminus X$ must contain a state in $\mathcal{F}_{\mathcal{B}\setminus X}$ that is an atom (in $\mathcal{F}_{\mathcal{B}\setminus X}$) at q; furthermore, this state is not an atom at r since $r \notin M \setminus X$. Since q and r were arbitrary, it follows that $\mathcal{F}_{\mathcal{B}\setminus X}$ is discriminative. □

As an immediate consequence of Theorem 4.11 and Lemma 4.8, we get the following characterization of the base of a partially $\cup$-closed and upper $\cap$-closed family.

Corollary 4.12. *Let $\mathcal{F}$ be a partially $\cup$-closed family with base $\mathcal{B}$, where $\cup\mathcal{F} = Q$. Then the following conditions are equivalent.*

(i) $\mathcal{F}$ is upper $\cap$-closed;

(ii) for any $X \in \mathcal{B}$, the space spanned by $\mathcal{B}\setminus X$ is both $\cup$-closed and $\cap$-closed; if $\mathcal{F}$ is discriminative, then the space spanned by $\mathcal{B}\setminus X$ is an ordinal knowledge space;

(iii) for any $X \in \mathcal{B}$, each item $q \in Q \setminus X$ has a unique atom in the space spanned by $\mathcal{B} \setminus X$.

From Theorem 4.11 we also get the following result.

Corollary 4.13. *Let $\mathcal{F}$ be a discriminative family of sets that is both partially $\cup$-closed and upper $\cap$-closed. Then $\mathcal{F}$ is well-graded.*

Proof. Let $X \in \mathcal{B}$, where $\mathcal{B}$ is the base of $\mathcal{F}$. By Theorem 4.11 the family spanned by $\mathcal{B} \setminus X$ is an ordinal space. Since an ordinal space is a special case of a learning space, by Lemma 4.6 it follows that $\mathcal{F}$ is well-graded. □

Example 4.14. Consider the following collection of sets:

$$\mathcal{B} = \{\{x, y, z\}, \{a, x\}, \{a, y\}, \{a, b, z\}, \{a, b\}\}.$$

The family spanned by $\mathcal{B}$ is partially $\cup$-closed, but it is not upper $\cap$-closed. To see this, note that $K = \{a, x, y, z\}$ and $L = \{a, b, x, z\}$ are both in the span of $\mathcal{B}$. However, $K \cap L = \{a, x, z\}$ is not in the span of $\mathcal{B}$, and since $\mathcal{B} \ni \{a, x\} \subset \{a, x, z\}$, it follows that the family spanned by $\mathcal{B}$ is not upper $\cap$-closed. Equivalently, applying Corollary 4.12 this can also be seen by noting that the family spanned by $\mathcal{B} \setminus \{a, x\}$ is not an ordinal knowledge space.

Based on the discussion in the previous paragraph (as well as a similar discussion for the sets $\{a, x, y, z\}$ and $\{a, b, y, z\}$), the family spanned by $\mathcal{B}$ is "missing" the states $\{a, x, z\}$ and $\{a, y, z\}$. Adding these states to $\mathcal{B}$ we get

$$\mathcal{B}' = \{\{x, y, z\}, \{a, x\}, \{a, y\}, \{a, b, z\}, \{a, b\}, \{a, x, z\}, \{a, y, z\}\}.$$

Note that the family spanned by $\mathcal{B}'$ is composed of everything in the span of $\mathcal{B}$, plus the two additional states $\{a, x, y, z\}$ and $\{a, b, y, z\}$. It is then straightforward to check that the family spanned by $\mathcal{B}'$ is both partially $\cup$-closed and upper $\cap$-closed.

5. Projections

As discussed in Section 1, projections of knowledge spaces have found applications in assessments of knowledge when it is problematic to run an assessment over a full knowledge space. In these situations, one option is to project the knowledge space on a suitably chosen subdomain.

The results of this section will use the terminology in the following two definitions from Chapter 2 of Falmagne and Doignon (2011). Suppose that $(Q, \mathcal{K})$ is a partial knowledge structure with $|Q| \geq 2$, and let Q' be any proper nonempty subset of Q.

Definition 5.1. Define a relation $\sim_{Q'}$ on $\mathcal{K}$ by

$$\begin{aligned} K \sim_{Q'} L &\iff K \cap Q' = L \cap Q' \\ &\iff K \triangle L \subseteq Q \setminus Q'. \end{aligned}$$

Note that $\sim_{Q'}$ (or, $\sim$, for short) is an equivalence relation on $\mathcal{K}$. We denote by $[K]$ the equivalence class of $\sim$ containing K, and by $\mathcal{K}_{\sim} = \{[K] \,|\, K \in \mathcal{K}\}$ the partition of $\mathcal{K}$ induced by $\sim$.

Definition 5.2. The family

$$\mathcal{K}_{|Q'} = \{W \subseteq Q' \,|\, W = K \cap Q' \text{ for some } K \in \mathcal{K}\}$$

is called the *projection* of K on Q'. Each set $W = K \cap Q'$ with $K \in \mathcal{K}$ is called the *trace* of the state K on Q'. For any state K in $\mathcal{K}$ and with $[K]$ as in Definition 5.1, we define the family

$$\mathcal{K}_{[K]} = \{M \subseteq Q \,|\, M = L \setminus \cap[K] \text{ for some } L \sim_{Q'} K\}.$$

The family $\mathcal{K}_{[K]}$ is called a *Q'-child*, or simply a *child* of $\mathcal{K}$ when the set Q' is made clear by the context.

Note that, as defined above, the idea of a projection has also found applications in media theory (Cavagnaro, 2008; Eppstein et al., 2008).

The following theorem is a variation of the Projection Theorem (Theorem 13 in Falmagne, 2008; Theorem 2.4.8 in Falmagne and Doignon, 2011); in addition to the assumption that the projection starts from a partially $\cup$-closed family, we will also assume that we have an upper $\cap$-closed family.

Theorem 5.3. *Let $\mathcal{F}$ be a well-graded partially $\cup$-closed family on a domain Q with $|Q| = |\cup \mathcal{F}| \geq 2$. Assume also that $\mathcal{F}$ is upper $\cap$-closed. The following two properties hold for any proper nonempty subset Q' of Q.*

(i) The projection $\mathcal{F}_{|Q'}$ of $\mathcal{F}$ on Q' is well-graded, partially $\cup$-closed, and upper $\cap$-closed. If $\mathcal{F}$ is an ordinal knowledge space (i.e., it contains the empty set), then so is $\mathcal{F}_{|Q'}$.

(ii) The above holds for the children of $\mathcal{F}$ as well. That is, the children of $\mathcal{F}$ are well-graded, partially $\cup$-closed, and upper $\cap$-closed. If $\mathcal{F}$ is an ordinal knowledge space, then so are the children of $\mathcal{F}$.

Proof.
(i) By (i) of Theorem 13 in Falmagne (2008), we have that $\mathcal{F}_{|Q'}$ is a well-graded partially $\cup$-closed family. To show that $\mathcal{F}_{|Q'}$ is upper $\cap$-closed, let $K, L \in \mathcal{F}_{|Q'}$. Then, for some $\widetilde{K}, \widetilde{L} \in \mathcal{F}$, we have $K = \widetilde{K} \cap Q'$ and $L = \widetilde{L} \cap Q'$. It follows that

$$\begin{aligned} K \cap L &= (\widetilde{K} \cap Q') \cap (\widetilde{L} \cap Q') \\ &= (\widetilde{K} \cap \widetilde{L}) \cap Q'. \end{aligned}$$

If $\mathcal{F}$ is an ordinal space, then $\widetilde{K} \cap \widetilde{L} \in \mathcal{F}$, from which it follows that $K \cap L \in \mathcal{F}_{|Q'}$. Also, $\varnothing \in \mathcal{F}$, which implies that $\varnothing \cap Q' = \varnothing \in \mathcal{F}_{|Q'}$. So, $\mathcal{F}_{|Q'}$ is an ordinal space.

Suppose next that $\mathcal{F}$ is not necessarily an ordinal space, but simply upper $\cap$-closed. In this case, some extra work is required to show that $\mathcal{F}_{|Q'}$ is upper $\cap$-closed as well. Suppose also that there is $B \subseteq K \cap L$ such that B is in the base of $\mathcal{F}_{|Q'}$. This means that, for some $q \in Q'$, B is an atom at q. Letting $\mathcal{B}$ be the base of $\mathcal{F}$, we claim that there exists $\widetilde{B} \in \mathcal{B}$ such that $\widetilde{B} \cap Q' = B$. To see this, consider any $\widetilde{C} \in \mathcal{F}$ where $B = \widetilde{C} \cap Q'$. Let $\widetilde{B}$ be an atom at q such that $\widetilde{B} \subseteq \widetilde{C}$. Thus, $\widetilde{B} \cap Q' \subseteq \widetilde{C} \cap Q' = B$, and since B is an atom at q in $\mathcal{F}_{|Q'}$, we must have $\widetilde{B} \cap Q' = B$.

Letting $\mathcal{F}_{\mathcal{B} \setminus \widetilde{B}}$ be the space spanned by $\mathcal{B} \setminus \widetilde{B}$, Corollary 4.12 tells us that $\mathcal{F}_{\mathcal{B} \setminus \widetilde{B}}$ is an ordinal space. We then have $(\widetilde{K} \setminus \widetilde{B}) \cap (\widetilde{L} \setminus \widetilde{B}) \in \mathcal{F}_{\mathcal{B} \setminus \widetilde{B}}$, and it follows that for some $A_i \in \mathcal{B}$, $i = 1, \ldots, n$,

$$(\widetilde{K} \setminus \widetilde{B}) \cap (\widetilde{L} \setminus \widetilde{B}) = \bigcup_{i=1}^{n} A_i \setminus \widetilde{B}.$$

Thus, we can define the set $\widetilde{M} = \bigcup_{i=1}^{n} A_i$, where $\widetilde{M} \in \mathcal{F}$. Furthermore, we have

$$\begin{aligned} \mathcal{F} \ni \widetilde{M} \cup \widetilde{B} &= \left(\bigcup_{i=1}^{n} A_i \right) \cup \widetilde{B} \\ &= \left(\bigcup_{i=1}^{n} A_i \setminus \widetilde{B} \right) \cup \widetilde{B} \\ &= \Big((\widetilde{K} \setminus \widetilde{B}) \cap (\widetilde{L} \setminus \widetilde{B}) \Big) \cup \widetilde{B} \\ &= \Big(\widetilde{K} \cap \widetilde{L} \Big) \cup \widetilde{B}. \end{aligned}$$

Projecting $\widetilde{M} \cup \widetilde{B}$ onto Q', we get

$$\begin{aligned}\left(\widetilde{M} \cup \widetilde{B}\right) \cap Q' &= \left(\left(\widetilde{K} \cap \widetilde{L}\right) \cup \widetilde{B}\right) \cap Q' \\ &= \left((\widetilde{K} \cap Q') \cap (\widetilde{L} \cap Q')\right) \cup \left(\widetilde{B} \cap Q'\right) \\ &= (K \cap L) \cup B \\ &= K \cap L,\end{aligned}$$

from which it follows that $K \cap L \in \mathcal{F}_{|Q'}$.

(ii) Let $K \in \mathcal{F}$, $Q' \subseteq Q$, and $L, M \in \mathcal{F}_{[K]}$. This means that there exists $\widetilde{L}, \widetilde{M} \in [K]$ such that $L = \widetilde{L} \setminus \cap[K]$ and $M = \widetilde{M} \setminus \cap[K]$. We have

$$\begin{aligned}L \cap M &= (\widetilde{L} \setminus \cap[K]) \cap (\widetilde{M} \setminus \cap[K]) \\ &= (\widetilde{L} \cap \widetilde{M}) \setminus \cap[K].\end{aligned}$$

If $\mathcal{F}$ is an ordinal space, $\widetilde{L} \cap \widetilde{M} \in \mathcal{F}$, from which it follows that $L \cap M \in \mathcal{F}_{[K]}$. Furthermore, note that $K \sim \cap[K]$ and, since $\mathcal{F}$ is $\cap$-closed, $\cap[K] \in \mathcal{F}$. It follows that $\varnothing = \cap[K] \setminus \cap[K] \in \mathcal{F}_{[K]}$, which shows that $\mathcal{F}_{[K]}$ is an ordinal space.

Next, assume $\mathcal{F}$ is only upper $\cap$-closed (and not necessarily an ordinal space). Suppose that there is $B \subseteq L \cap M$ such that B is in the base of $\mathcal{F}_{[K]}$. Let $\widetilde{B}$ be such that $\widetilde{B} \sim K$ and $B = \widetilde{B} \setminus \cap[K]$. We claim that $\widetilde{B} \subseteq \widetilde{L} \cap \widetilde{M}$. Let $q \in \widetilde{B}$. If $q \in \cap[K]$, then since $\widetilde{L}, \widetilde{M} \in [K]$, we have $q \in \widetilde{L} \cap \widetilde{M}$. Next, suppose $q \in \widetilde{B} \setminus \cap[K] = B$. The result follows since $B \subseteq L \cap M \subseteq \widetilde{L} \cap \widetilde{M}$.

Having shown that $\widetilde{B} \subseteq \widetilde{L} \cap \widetilde{M}$, it is clear that $\widetilde{L} \cap \widetilde{M}$ contains some $A \in \mathcal{B}$, which by upper $\cap$-closure implies that $\widetilde{L} \cap \widetilde{M} \in \mathcal{F}$. Thus, $\mathcal{F}_{[K]}$ is upper $\cap$-closed. □

6. Infinite families of sets

All the results thus far have assumed a finite family of sets on a finite domain of items. In this section we will extend some of the results from Section 4 to infinite families of sets. In what follows, we will avoid extra technical complications by assuming that all families of sets are discriminative. To start, we will use the following definitions from Falmagne and Doignon (2011) that extend the concept of well-gradedness to infinite families.

Definition 6.1. Given a family of (possibly infinite) sets $\mathcal{F}$, a *chain* in the partially ordered set $(\mathcal{F}, \subseteq)$ is any subset $\mathcal{G}$ of $\mathcal{F}$ such that $A \subseteq B$ or $B \subseteq A$ for any $A, B \in \mathcal{G}$. A *learning path* is a maximal chain. In the infinite case, the existence of such a maximal chain is established by applying the Hausdorff maximal principle (for a statement and proof of the maximal principle see, for example, Section 1-11 in Munkres, 1975).

Definition 6.2. Let $\mathcal{F}$ be a family of sets. A subfamily $\mathcal{D}$ of sets is a *bounded path* connecting sets K and L if it contains K and L and the following three conditions hold: for all distinct D and E in $\mathcal{D}$,

(1) $\qquad K \cap L \subseteq D \subseteq K \cup L;$

(2) either $\quad D \setminus L \subseteq E \setminus L$ and $D \setminus K \supseteq E \setminus K,$

or $\quad D \setminus L \supseteq E \setminus L$ and $D \setminus K \subseteq E \setminus K;$

(3) either (a) $\exists F \in \mathcal{D} \setminus \{D\}, \exists q \in D \setminus F : F \cup \{q\} = D,$

or (b) $\begin{cases} D \setminus K = \cup\{G \setminus K \,|\, G \in \mathcal{D}, G \setminus K \subset D \setminus K\}, \\ \text{and} \\ D \setminus L = \cup\{G \setminus L \,|\, G \in \mathcal{D}, G \setminus L \subset D \setminus L\}. \end{cases}$

A family of sets is ∞-*well-graded* if any two of its sets are connected by a bounded path.

We will also need the following definition, which has been modified from its original form in Falmagne and Doignon (2011) to account for the lack of an empty set.

Definition 6.3. Let $\mathcal{C}$ be a learning path in a family of sets $\mathcal{F}$, and let $M = \bigcap_{C \in \mathcal{C}} C$. Then $\mathcal{C}$ is an ∞-*gradation* if for any $K \in \mathcal{C} \setminus \{M\}$ we have the following:

$$\text{either} \qquad K = K' \cup \{q\}, \text{ for some } q \in K \text{ and } K' \in \mathcal{C} \setminus \{K\}, \tag{6.1}$$

$$\text{or} \qquad K = \cup\{L \in \mathcal{C} \,|\, L \subset K\}. \tag{6.2}$$

Note that it is possible to have $M = \varnothing$ or $M \notin \mathcal{C}$.

The following is an example of a partially $\cup$-closed and upper $\cap$-closed family that is also ∞-well-graded.

Example 6.4. Let the family $\mathcal{F}$ consist of the span of all sets of the form

$$A_n = \{z \in \mathbb{Z} \,|\, z \leq n\}, n \in \mathbb{Z}.$$

For any $n, m \in \mathbb{Z}$ with $n < m$ we have

$$A_n \cap A_m = A_n$$

and

$$A_n \cup A_m = A_m.$$

Furthermore, we also have

$$\mathbb{Z} = \left(\bigcup_{n \in \mathbb{Z}} A_n \right) \in \mathcal{F}.$$

Thus, since it is easy to see that $\varnothing \notin \mathcal{F}$, it follows that $\mathcal{F}$ is both partially $\cup$-closed and upper $\cap$-closed.

Next, note that the sets $A_n, A_{n+1}, \ldots, A_{m-1}, A_m$ form a bounded path from A_n to A_m. Also, for any $A_n \in \mathcal{F}$, $D = \{A_i \,|\, i \geq n\}$ is a bounded path from A_n to $\mathbb{Z}$. Thus, it follows that $\mathcal{F}$ is also ∞-well-graded.

We will also make use of the following result from Section 4.3 in Falmagne and Doignon (2011).

Theorem 6.5 (Falmagne and Doignon)**.** *For any discriminative partial knowledge space $(Q, \mathcal{K})$, the following two conditions are equivalent:*

(i) $(Q, \mathcal{K})$ *is ∞-well-graded;*

(ii) *all the learning paths in $(Q, \mathcal{K})$ are ∞-gradations.*

Note that the above result is formulated in Falmagne and Doignon (2011) for knowledge spaces. However, the arguments used in the proof carry over to partial knowledge spaces, and it is the above version we will need in what follows.

Our main result is a version of Lemma 4.6 for infinite families which, due to the extra complexity of the proof, will be formulated as Theorem 6.6.

Theorem 6.6. *Let $\mathcal{B}$ be the base of a partially $\cup$-closed family $\mathcal{F}$. Then $\mathcal{F}$ is ∞-well-graded if and only if, for each X in $\mathcal{B}$, the family spanned by $\mathcal{B} \setminus X$ is ∞-well-graded.*

Proof.
$\mathcal{F}$ is ∞-well-graded $\Rightarrow$ for every $X \in \mathcal{B}$, the span of $\mathcal{B} \setminus X$ is ∞-well-graded:

For some $X \in \mathcal{B}$ let $\mathcal{F}_{\mathcal{B}\setminus X}$ be the family spanned by $\mathcal{B} \setminus X$. Let $\mathcal{C}$ be a learning path in $\mathcal{F}_{\mathcal{B}\setminus X}$, and let $K \in \mathcal{C} \setminus \{M\}$ (where M is defined as in Definition 6.3). If we can show that $\mathcal{C}$ is an ∞-gradation, by Theorem 6.5 it will follow that $\mathcal{F}_{\mathcal{B}\setminus X}$ is ∞-well-graded. Define the set $N = \cup\{L \in \mathcal{C} \,|\, L \subset K\}$. Note that by the definition of N, for any $D \in \mathcal{C}$ with $N \subseteq D \subseteq K$, we must have either $N = D$ or $D = K$. In order to show that either 6.1 or 6.2 holds, we must have $|K \setminus N| \leq 1$. We will proceed by contradiction. That is, assume $|K \setminus N| \geq 2$.

By hypothesis, $\mathcal{F}$ is ∞-well-graded; thus, there exists a bounded path in $\mathcal{F}$, $\widetilde{\mathcal{D}}$, from $N \cup X$ to $K \cup X$. Since $|K \setminus N| \geq 2$, there exists $\widetilde{D} \in \widetilde{\mathcal{D}}$ such that $N \cup X \subset \widetilde{D} \subset K \cup X$. However, this implies that $\widetilde{D} \setminus X \in \mathcal{F}$, where $N \subset \widetilde{D} \setminus X \subset K$. Thus, since there are no sets in $\mathcal{C}$ that lie between N and K, it follows that $\widetilde{D} \setminus X$ is totally ordered with respect to $\mathcal{C}$. However, this means that $\mathcal{C} \cup (\widetilde{\mathcal{D}} \setminus X)$ is also a chain, contradicting the assumption that $\mathcal{C}$ is a maximal chain. So, it must be the case that $|K \setminus N| \leq 1$.

For every $X \in \mathcal{B}$, the span of $\mathcal{B} \setminus X$ is ∞-well-graded $\Rightarrow$ $\mathcal{F}$ is ∞-well-graded:

Let $\widetilde{K}, \widetilde{L} \in \mathcal{F}$, and let $X \in \mathcal{B}$ with $X \subseteq \widetilde{K}$. Then, $K = \widetilde{K} \setminus X \in \mathcal{F}_{\mathcal{B}\setminus X}$ and $L = \widetilde{L} \setminus X \in \mathcal{F}_{\mathcal{B}\setminus X}$, where $\mathcal{F}_{\mathcal{B}\setminus X}$ is the family spanned by $\mathcal{B} \setminus X$. By hypothesis, there exists a bounded path $\mathcal{D} \subseteq \mathcal{F}_{\mathcal{B}\setminus X}$ from K to $K \cup L$. Consider the path $\widetilde{\mathcal{D}} \subseteq \mathcal{F}$, where $\widetilde{\mathcal{D}} = \{D \cup X \,|\, D \in \mathcal{D}\}$. We claim that $\widetilde{\mathcal{D}}$ is a bounded path from $K \cup X = \widetilde{K}$ to $(K \cup X) \cup (L \cup X) = \widetilde{K} \cup \widetilde{L}$.

To start, note that since $K, K \cup L \in \mathcal{D}$, we have $\widetilde{K} = K \cup X \in \widetilde{\mathcal{D}}$ and $\widetilde{K} \cup \widetilde{L} = (K \cup L) \cup X \in \mathcal{D}$. Since $\mathcal{D}$ is a bounded path from K to $K \cup L$, (1) and (2) from Definition 6.2 hold. It is straightforward to see that these conditions still hold for $\widetilde{\mathcal{D}}$. That is, taking the union of every set in $\mathcal{D}$ with X preserves all the set orderings in (1) and (2). So, if we can show that (3) holds, it will follow that $\widetilde{\mathcal{D}}$ is a bounded path from $\widetilde{K}$ to $\widetilde{K} \cup \widetilde{L}$.

Suppose (3)(a) holds for $\mathcal{D}$. Then, $\exists\, F \in \mathcal{D} \setminus \{D\}, \exists\, q \in D \setminus F : F \cup \{q\} = D$. It follows that $F \cup X \in \widetilde{\mathcal{D}} \setminus \{D \cup X\}$, where $F \cup X \cup \{q\} = D \cup X$. Thus (3)(a) also holds for $\widetilde{\mathcal{D}}$.

Suppose (3)(b) holds for $\mathcal{D}$. We will first note that $(D \cup X) \setminus (K \cup X) = D \setminus K$ and $(D \cup X) \setminus (K \cup L \cup X) = D \setminus (K \cup L)$. Next, for any $G \in \mathcal{D}$ we have

$$G \setminus K \subset D \setminus K \Rightarrow (G \cup X) \setminus (K \cup X) \subset (D \cup X) \setminus (K \cup X).$$

Similarly, we also have

$$G \setminus (K \cup L) \subset D \setminus (K \cup L) \Rightarrow \\ (G \cup X) \setminus (K \cup L \cup X) \subset (D \cup X) \setminus (K \cup L \cup X).$$

Thus, since $(G \cup X) \setminus (K \cup X) = G \setminus K$ for any $G \in \mathcal{D}$, it follows that

$$\cup \{G \setminus K \,|\, G \in \mathcal{D}, G \setminus K \subset D \setminus K\} = \\ \cup \{(G \cup X) \setminus (K \cup X) \,|\, G \cup X \in \widetilde{\mathcal{D}}, (G \cup X) \setminus (K \cup X) \subset (D \cup X) \setminus (K \cup X)\}.$$

Applying the same argument for $K \cup L$, we get that $\widetilde{\mathcal{D}}$ is a bounded path from $\widetilde{K} = K \cup X$ to $\widetilde{K} \cup \widetilde{L} = K \cup L \cup X$.

To finish the proof, notice that we can apply the preceding argument to get a bounded path from $\widetilde{L}$ to $\widetilde{K} \cup \widetilde{L}$ as well. Taking the union of these two paths then gives a bounded path from $\widetilde{K}$ to $\widetilde{L}$.

□

As a corollary of Theorem 6.6, we get the following version of Theorem 4.7 for infinite families.

Corollary 6.7. *Let $\mathcal{F}$ be a partially $\cup$-closed family with base $\mathcal{B}$. Let $\sigma_{\mathcal{B}\setminus X}$ be the surmise function of the space spanned by $\mathcal{B} \setminus X$. Then the following conditions are equivalent:*

(i) $\mathcal{F}$ is ∞-well-graded;

(ii) for any $X \in \mathcal{B}$, the family $\{\sigma_{\mathcal{B}\setminus X} \,|\, x \in \cup\mathcal{B} \setminus X\}$ is a partition of $\mathcal{B} \setminus X$;

(iii) for any $X \in \mathcal{B}$ the following holds: for any atom A at $q \in \cup\mathcal{B} \setminus X$ in the space spanned by $\mathcal{B} \setminus X$, the set $A \setminus \{q\}$ is a state.

7. Discussion

In this paper we have explored various properties of well-graded partially $\cup$-closed families. We began by taking two axioms for learning spaces and modifying them to apply in the more general case of families that do not contain the empty set. We then showed that several results and theorems for well-graded $\cup$-closed families have analogues for partially $\cup$-closed families. As part of this work, we have discussed possible solutions to two open problems given in Falmagne and Doignon (2011), and we have also looked at how the concept of being $\cap$-closed extends to families without the empty set.

8. Acknowledgements

The author would like to express his gratitude to Chris Doble, Eric Cosyn and Hasan Uzun, whose insightful comments shaped and improved several important parts of the exposition. The author would also like to thank Jean-Paul Doignon and the two referees for their thoughtful and detailed suggestions. In particular, both the statement and proof of Theorem 3.3 are due to Jean-Paul Doignon who, upon reviewing this material, graciously shared these results with the author.

References

Cavagnaro, D., 2008. Projection of a medium. Journal of Mathematical Psychology 52, 55–63.

Cosyn, E., Uzun, H., 2009. Note on two sufficient axioms for a well-graded knowledge space. Journal of Mathematical Psychology 53, 40–42.

Doignon, J.P., Falmagne, J.C., 1985. Spaces for the assessment of knowledge. International Journal of Man-Machine Studies 23, 175–196.

Doignon, J.P., Falmagne, J.C., 1997. Well-graded families of relations. Discrete mathematics 173, 35–44.

Eppstein, D., Falmagne, J.C., Ovchinnikov, S., 2008. Media Theory. Springer-Verlag, Berlin, Heidelberg, and New York.

Eppstein, D., Falmagne, J.C., Uzun, H., 2009. On verifying and engineering the well-gradedness of a union-closed family. Journal of Mathematical Psychology 53, 34–39.

Falmagne, J.C., 2008. Projections of a learning space. Electronic preprint 0803.0575v2, arXiv.org.

Falmagne, J.C., Albert, D., Doble, C., Eppstein, D., Hu, X. (Eds.), 2013. Knowledge Spaces: Applications in Education. Springer-Verlag, Heidelberg.

Falmagne, J.C., Doignon, J.P., 2011. Learning Spaces. Springer-Verlag, Heidelberg.

Koppen, M., 1998. On alternative representations for knowledge spaces. Mathematical Social Sciences 36, 127–143.

Munkres, J., 1975. Topology: A First Course. Prentice Hall, Englewood Cliffs, New Jersey.