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AN APPLICATION OF RANDOM POLYNOMIALS TO WIRELESS COMMUNICATIONS

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ABSTRACT. Shepp and Vanderbei developed techniques for analyzing the complex zeros of a random polynomial with independent standard normal coefficients. In this paper we will adapt their techniques and apply them to a problem posed by Schober and Gerstacker concerning the GSM (Global System for Mobile Communications)/EDGE (Enhanced Data Rates for GSM Evolution) standard for mobile phones. The problem is to study the behavior of the complex roots of random polynomials with mean zero complex Gaussian coefficients, where the variances are exponentially increasing or decreasing. While Schober and Gerstacker studied the case of independent coefficients, we will look at what happens when they are assumed to be dependent. Applying a result of Hughes and Nikeghbali, we will first show that, without any restrictions on the dependence of the coefficients, the roots accumulate around a circle in the complex plane, uniformly in the angle, where the radius is determined by the coefficient variances. This result matches that obtained in the independent case. By adding certain conditions to the covariance function of the coefficients, we will then be able to use Shepp and Vanderbei's techniques to obtain a more detailed analysis of this behavior.

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1. INTRODUCTION

Consider the random polynomial given by

$$P_n(z) = \sum_{k=0}^{n} Z_k z^k. \tag{1.1}$$

For independent standard normal coefficients, Shepp and Vanderbei [12] derived a formula for computing the expected number of zeros in a given subset of the complex plane. Additionally, they showed that the zeros tend to accumulate around the unit circle, uniformly in the angle. More recently, Hughes and Nikeghbali [7] extended this result under much more general assumptions on the coefficients of (1.1). In this paper we will focus on a problem concerning the GSM (Global System for Mobile

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Communications)/EDGE (Enhanced Data Rates for GSM Evolution) standard for mobile phones.

When designing digital receivers for such a system, the properties of the so-called discrete-time overall channel impulse response becomes important. Specifically, the location of the roots of the z-transform of the discrete-time overall channel impulse response determines the receiver's performance. The randomness inherent in mobile communications results in such a z-transform being a random polynomial. For wireless communications in urban areas it is common for the coefficients of (1.1) to be mean zero complex Gaussians, with exponentially increasing or decreasing variances (see [11] and the references therein for a more complete discussion). Under these assumptions, Schober and Gerstacker derived explicit results for the roots' location when the coefficients are independent. This assumption of independence, however, was made to facilitate the computations. In practice, the authors state that the coefficients will only be approximately uncorrelated.

With that in mind, this paper's goal is to study the behavior of the complex roots when the coefficients are dependent mean zero complex Gaussians with exponentially increasing or decreasing variances. Using a result from Hughes and Nikeghbali, we will first show that, in the limit, the roots accumulate around a circle in the complex plane, uniformly in the angle, where the radius is determined by the coefficient variances. This behavior holds without any restrictions on the covariance function of the coefficients and corresponds with the behavior observed by Schober and Gerstacker in the independent case. The drawback is that this result applies only to the limiting behavior, and it fails to give any detail as to how fast this occurs or how close to the circle the zeros accumulate. Thus, to get a more detailed analysis we will use the techniques developed by Shepp and Vanderbei. In order for us to apply these techniques when the coefficients are dependent, some concessions must be made. Namely, it will be necessary for us to assume that the covariance function of the coefficients is absolutely summable and that the spectral density (which will be introduced in Section 3) does not vanish. Another way to interpret these conditions is that we are requiring fast enough decay for the covariance of the coefficients.

2. GENERAL BEHAVIOR OF THE COMPLEX ZEROS

We will start by giving a result from Hughes and Nikeghbali [7]. Let $P_n(z)$ be of the form given in (1.1), and let $\nu_n(\Omega)$ be the number of zeros of $P_n(z)$ in the set Ω. Also, for $0 < r < 1$ define the annulus $a(r) = \{z \in \mathbb{C} : 1 - r \le |z| \le 1/(1 - r)\}$, and for $0 \le \theta_1 < \theta_2 \le 2\pi$ let $C(\theta_1, \theta_2)$ be the cone in the complex plane consisting of all points with arguments between θ_1 and θ_2.

Theorem 2.1 (Hughes and Nikeghbali). *Assume the coefficients of $P_n(z)$ are complex Gaussians with mean zero and unit variance. Then there exists a deterministic positive sequence (α_n), subject to $0 < \alpha_n \leq n$ for all n and $\alpha_n = o(n)$ as $n \to \infty$, such that*

$$\lim_{n\to\infty} \frac{1}{n}\nu_n\left(a\left(\frac{\alpha_n}{n}\right)\right) = 1, \quad a.s.$$

and

$$\lim_{n\to\infty} \frac{1}{n}\nu_n\left(C(\theta_1,\theta_2)\right) = \frac{\theta_2-\theta_1}{2\pi}, \quad a.s.$$

In other words, the above theorem tells us that for mean zero complex Gaussian coefficients with unit variance, the roots will accumulate around the unit circle in the limit, uniformly in the angle. Furthermore, this occurs without any restrictions on the dependence of the coefficients. Now, consider the random polynomial

$$\tilde{P}_n(z) = \sum_{k=0}^{n} \sigma e^{\beta(n-k)/2} Z_k z^k,$$

where the Z_k are mean zero complex Gaussians with unit variance, $\sigma > 0$, and $\beta \in \mathbb{R}$. Thus, the coefficients now have exponentially growing or decaying variances, depending on the value of β. Let z_0 be a root of $P_n(z)$. Then,

$$\begin{aligned}\tilde{P}_n(e^{\beta/2}z_0) &= \sigma\left(e^{\beta n/2}Z_0 + e^{\beta(n-1)/2}Z_1 e^{\beta/2}z_0 + \cdots + Z_n\left(e^{\beta/2}z_0\right)^n\right)\\ &= \sigma e^{\beta n/2}\left(Z_0 + Z_1 z_0 + \cdots + Z_n z_0^n\right)\\ &= 0,\end{aligned}$$

and it follows that $e^{\beta/2}z_0$ is a root of $\tilde{P}_n(z)$. Applying Theorem 2.1, we can then conclude that the roots of $\tilde{P}_n(z)$ accumulate around a circle of radius $e^{\beta/2}$, uniformly in the angle. Furthermore, the fact that the expected number of roots of $P_n(z)$ inside the unit circle is equal to the expected number outside implies the same property for $\tilde{P}_n(z)$ and the circle of radius $e^{\beta/2}$.

To summarize, when the coefficients are dependent complex Gaussians with mean zero and exponentially increasing or decreasing variances, we have shown that the zeros will accumulate around the circle of radius $e^{\beta/2}$ in the limit. Additionally, they will do so uniformly in the angle, and the expected number of roots inside the circle will be equal to the expected number outside. The rest of this paper's goal will be to give a more thorough analysis of this behavior. This will be accomplished by imposing some restrictions on the covariance function of the coefficients, which will then allow us to use Shepp and Vanderbei's techniques to give this more detailed discussion.

3. MAIN RESULT

We will begin with a comment about the covariance function of the coefficients. Assuming Gaussian coefficients, we can follow the procedure from [8, 9] and express the covariance function, $\Gamma(k)$, as

$$\mathrm{E}[X_0 X_k] = \Gamma(k) = \int_{-\pi}^{\pi} e^{-ik\phi} f(\phi) d\phi, \tag{3.1}$$

where $f(\phi)$ is the spectral density of the covariance function (see also [2, 4] for further reference). A sufficient condition for the existence of $f(\phi)$ is that $\Gamma(k)$ be absolutely summable. Additionally, in this case it will be nonnegative, continuous, and of the form

$$f(\phi) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \Gamma(k) e^{ik\phi}.$$

Throughout the rest of this discussion we will let $\Gamma(k)$ be the covariance function of a sequence of dependent standard normal random variables, and we will assume that $P_n(z)$ has the form

$$P_n(z) = \sum_{k=0}^{n} (U_k + iV_k) z^k = \sum_{k=0}^{n} Z_k z^k, \tag{3.2}$$

where the coefficients are complex Gaussians with mean zero. In addition, they will have exponentially increasing or decreasing variances; that is,

$$\mathrm{E}\left[Z_k \overline{Z}_k\right] = \sigma_k^2 = \sigma^2 e^{\beta(n-k)}, \tag{3.3}$$

for $0 \leq k \leq n$, $\sigma > 0$, and $\beta \in \mathbb{R}$. In [11] the coefficients were taken to be independent to simplify the calculations. We will now assume some dependence among the coefficients, where the covariance is given by

$$\begin{aligned} \mathrm{E}\left[Z_k \overline{Z}_j\right] &= \mathrm{E}\left[(U_k + iV_k)(U_j - iV_j)\right] \\ &= \mathrm{E}\left[U_k U_j\right] + \mathrm{E}\left[V_k V_j\right] \\ &= \frac{\sigma^2 e^{\beta(2n-k-j)/2}}{2} \Gamma(k-j) + \frac{\sigma^2 e^{\beta(2n-k-j)/2}}{2} \Gamma(k-j) \\ &= \sigma^2 e^{\beta(2n-k-j)/2} \Gamma(k-j). \end{aligned} \tag{3.4}$$

Thus,

$$\mathrm{E}\left[U_k U_j\right] = \mathrm{E}\left[V_k V_j\right] = \frac{1}{2} \mathrm{E}\left[Z_k \overline{Z}_j\right]. \tag{3.5}$$

Two additional expressions that we will need are

$$\begin{aligned} B_0(z) &= \mathrm{E}\left[P_n(z) \overline{P_n(z)}\right], \\ B_1(z) &= \mathrm{E}\left[P_n(z) \overline{z} \overline{P_n'(z)}\right]. \end{aligned} \tag{3.6}$$

One main difference from the independent case is that these expressions are not straightforward to compute; they depend on the values of the spectral density. To apply these formulas we will rely heavily on deriving asymptotic values throughout this paper. As before, let $\nu_n(\Omega)$ be the number of zeros of $P_n(z)$ in the set Ω. We are now ready to state our first theorem, which extends Shepp and Vanderbei's result to our particular case.

Theorem 3.1. *For any region $\Omega \in \mathbb{C}$ whose boundary intersects the real axis at most finitely many times we have*

$$\mathrm{E}[\nu_n(\Omega)] = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{1}{z}F(z)dz, \tag{3.7}$$

where

$$F(z) = \frac{\overline{B_1(z)}}{B_0(z)}. \tag{3.8}$$

Proof. As noted by the authors in [12], the proof used for real Gaussians can be applied to complex Gaussians, and in which case the computations will simplify. The first part of this proof will carry out these simplified calculations, while the second part will apply the spectral density form of the covariance function to compute the needed expressions.

To start, we can use the argument principle to compute $\nu_n(\Omega)$. It follows that

$$\nu_n(\Omega) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{P_n'(z)}{P_n(z)}dz.$$

By applying Fubini's Theorem and a result of Hammersley [6] on the distribution of the zeros of a random polynomial with complex Gaussian coefficients, we can take the expectation and move it inside the integral. Thus, we arrive at the formula

$$\begin{aligned}\mathrm{E}[\nu_n(\Omega)] &= \frac{1}{2\pi i}\int_{\partial\Omega} \mathrm{E}\left[\frac{P_n'(z)}{P_n(z)}\right]dz \\ &= \frac{1}{2\pi i}\int_{\partial\Omega} \frac{1}{z}\mathrm{E}\left[\frac{zP_n'(z)}{P_n(z)}\right]dz.\end{aligned}$$

We will now derive the formula for $F(z) = \mathrm{E}\left[\frac{zP_n'(z)}{P_n(z)}\right]$ given in (3.8). To do this, we note that, for a fixed z, the vector $(P_n(z), P_n'(z))$ is a complex Gaussian with mean zero. Furthermore, the covariance matrix is given by

$$\begin{bmatrix} B_0(z) & \frac{1}{z}B_1(z) \\ \frac{1}{z}\overline{B_1(z)} & \frac{1}{z}B_1'(z) \end{bmatrix}. \tag{3.9}$$

So, letting $\alpha = \frac{\overline{B_1(z)}}{zB_0(z)}$ and $\beta = \frac{1}{z}B_1'(z) - \alpha B_0(z)$, it follows that $P_n'(z) =_d \alpha P_n(z) + \beta U$, where U is a standard complex Gaussian that is independent of $P_n(z)$. We then have

$$\begin{aligned}F(z) &= \mathrm{E}\left[\frac{zP_n'(z)}{P_n(z)}\right] \\ &= \mathrm{E}\left[\frac{z\left(\alpha P_n(z) + \beta U\right)}{P_n(z)}\right] \\ &= z\alpha + \mathrm{E}\left[\frac{z\beta U}{P_n(z)}\right]\end{aligned}$$

$$= \frac{\overline{B_1(z)}}{B_0(z)},$$

as claimed. □

4. APPLICATIONS

Once we have verified Shepp and Vanderbei's formula for the expected number of zeros when some dependence is assumed among the coefficients, we can discuss some applications. We will proceed as they did, proving a couple of results which illustrate the behavior of the complex roots. While we are expecting similar behavior as in the independent case, the extra assumption of dependence will force us to rely on the spectral density form of the covariance function, along with several asymptotic results, to show this. We will prove two theorems that give a more detailed description of the accumulation of roots around the circle of radius $e^{\beta/2}$.

Theorem 4.1. *Let $D(r)$ be the disk of radius r centered at 0. For any $s \geq 0$ we have*

$$\begin{aligned} \mathrm{E}\left[\nu_n\left(D\left(e^{\beta/2-s/2(n+1)}\right)\right)\right] &\sim \frac{-(n+1)e^{-s}}{1-e^{-s}} + \frac{e^{-s/(n+1)}}{1-e^{-s/(n+1)}} \\ &\sim (n+1)\frac{1-e^{-s}(1+s)}{s(1-e^{-s})}, \end{aligned}$$

as $n \to \infty$. Note that the first line is an equality in the independent case. Letting $s \to 0$, it follows that

$$\sim (n+1)\left(\frac{1}{2} - \frac{s}{3}\right).$$

Proof. From (3.7) we have

$$\begin{aligned} \mathrm{E}[\nu_n(D(r))] &= \frac{1}{2\pi i}\int_{\partial D(r)} \frac{1}{z}F(z)dz \\ &= \frac{1}{2\pi}\int_0^{2\pi} F(re^{i\theta})d\theta, \end{aligned} \tag{4.1}$$

where

$$r = e^{\beta/2-s/2(n+1)}, \quad s \geq 0, \quad z = re^{i\theta},$$

and F is as in (3.8). We will need to determine the asymptotic behavior of $\overline{B_1(z)}$ and $B_0(z)$. Note that we can assume θ is bounded some small distance away from $-\pi$ and π. Otherwise, using the fact that

$$\Gamma(k) = \int_{-\pi}^{\pi} e^{-ik\phi}f(\phi)d\phi = \int_0^{2\pi} e^{-ik\phi}f(\phi)d\phi = \int_{-2\pi}^{0} e^{-ik\phi}f(\phi)d\phi$$

for any k, the following results will hold with only minor changes to the arguments used.

Starting first with $B_0(z)$ we have

$$\begin{aligned}
B_0(z) &= \sigma^2 e^{\beta n}\int_{-\pi}^{\pi} f(\phi)\frac{1-e^{-s/2}e^{i(n+1)(\theta-\phi)}}{1-e^{-s/2(n+1)}e^{i(\theta-\phi)}}\cdot\frac{1-e^{-s/2}e^{i(n+1)(\phi-\theta)}}{1-e^{-s/2(n+1)}e^{i(\phi-\theta)}}d\phi \\
&= \sigma^2 e^{\beta n}\int_{\theta-(n+1)^{-\frac14}}^{\theta+(n+1)^{-\frac14}} f(\phi)\frac{1-2e^{-s/2}\cos\left[(n+1)(\theta-\phi)\right]+e^{-s}}{1-2e^{-s/2(n+1)}\cos\left(\theta-\phi\right)+e^{-s/(n+1)}}d\phi \\
&\quad + \sigma^2 e^{\beta n}\int_{\theta+(n+1)^{-\frac14}}^{\pi} f(\phi)\frac{1-2e^{-s/2}\cos\left[(n+1)(\theta-\phi)\right]+e^{-s}}{1-2e^{-s/2(n+1)}\cos\left(\theta-\phi\right)+e^{-s/(n+1)}}d\phi \\
&\quad + \sigma^2 e^{\beta n}\int_{-\pi}^{\theta-(n+1)^{-\frac14}} f(\phi)\frac{1-2e^{-s/2}\cos\left[(n+1)(\theta-\phi)\right]+e^{-s}}{1-2e^{-s/2(n+1)}\cos\left(\theta-\phi\right)+e^{-s/(n+1)}}d\phi \\
&= B_0^1+B_0^2+B_0^3.
\end{aligned}$$

For B_0^1 we have,

$$\begin{aligned}
B_0^1 &\sim 2\sigma^2 e^{\beta n}\int_{\theta}^{\theta+(n+1)^{-\frac14}} c_n f(\phi) \\
&\quad\cdot\left(\frac{1}{2-2(1-\frac{s}{2(n+1)}+\frac{s^2}{8(n+1)^2})(1-\frac{(\theta-\phi)^2}{2})-\frac{s}{n+1}+\frac{s^2}{2(n+1)^2}}\right)d\phi \\
&\sim 2c_n\sigma^2 e^{\beta n} f(\theta)\int_{\theta}^{\theta+(n+1)^{-\frac14}}\frac{d\phi}{(\theta-\phi)^2+\frac{s^2}{4(n+1)^2}} \\
&= c_n\sigma^2 e^{\beta n} f(\theta)\frac{4}{s}(n+1)\arctan\left(\frac{2}{s}(n+1)(\phi-\theta)\right)\Big|_{\theta}^{\theta+(n+1)^{-1/4}} \\
&\sim c_n\sigma^2 e^{\beta n} f(\theta)\frac{2\pi}{s}(n+1).
\end{aligned}$$

We will next show that B_0^2 and B_0^3 are small compared to B_0^1. For B_0^2,

$$\begin{aligned}
B_0^2 &\sim \sigma^2 e^{\beta n}\int_{\theta+(n+1)^{-\frac14}}^{\pi}\frac{cf(\phi)}{1-2e^{-s/2(n+1)}\cos\left(\theta-\phi\right)+e^{-s/(n+1)}}d\phi \\
&\le \sigma^2 e^{\beta n}\int_{\theta+(n+1)^{-\frac14}}^{\pi}\frac{cf(\phi)}{1-2e^{-s/2(n+1)}\cos\left(-(n+1)^{-1/4}\right)+e^{-s/(n+1)}}d\phi \\
&\sim \sigma^2 e^{\beta n}\int_{\theta+(n+1)^{-\frac14}}^{\pi}\frac{cf(\phi)}{(n+1)^{-1/2}+\frac{s^2}{4(n+1)^2}}d\phi \\
&\sim \sigma^2 e^{\beta n} c(n+1)^{1/2} \\
&= o\left(B_0^1\right).
\end{aligned}$$

Similarly, we can also show that $B_0^3 = o\left(B_0^1\right)$. It follows that

$$B_0(z)\sim B_0^1\sim c_n\sigma^2 e^{\beta n} f(\theta)\frac{2\pi}{s}(n+1).$$

In the independent case $f(\theta) \equiv \frac{1}{2\pi}$. Setting the quantity above equal to the value of $B_0(z)$ in the independent case, $\sigma^2 e^{\beta n} \frac{1-e^{-s}}{1-e^{-s/(n+1)}}$, allows us to solve for c_n. Thus,

$$\sigma^2 e^{\beta n}(n+1)\frac{c_n}{s} \sim \sigma^2 e^{\beta n}\frac{1-e^{-s}}{1-e^{-s/(n+1)}} \Rightarrow c_n = \frac{s}{n+1} \cdot \frac{1-e^{-s}}{1-e^{-s/(n+1)}},$$

and we have now shown that

$$B_0(z) \sim 2\pi\sigma^2 e^{\beta n}\frac{1-e^{-s}}{1-e^{-s/(n+1)}} f(\theta). \tag{4.2}$$

Next, for $B_1(z)$ we have

$$\begin{aligned}
B_1(z) = \sigma^2 e^{\beta n} \int_{-\pi}^{\pi} \Bigg[& \frac{\left((ze^{-\beta/2})^{n+1} e^{-i(n+1)\phi} - 1\right)(n+1)\left(\overline{z}e^{-\beta/2}e^{i\phi}\right)^{n+1}}{\left(1 - ze^{-\beta/2}e^{-i\phi}\right)\left(1 - \overline{z}e^{-\beta/2}e^{i\phi}\right)} \\
& + \frac{\left|1 - \left(ze^{-\beta/2}\right)^{n+1} e^{-i(n+1)\phi}\right|^2 \left(\overline{z}e^{-\beta/2}e^{i\phi} - |z|^2 e^{-\beta}\right)}{\left(1 - ze^{-\beta/2}e^{-i\phi}\right)^2 \left(1 - \overline{z}e^{-\beta/2}e^{i\phi}\right)^2} \Bigg] f(\phi) d\phi \\
= \sigma^2 e^{\beta n} \int_{-\pi}^{\pi} f(\phi) \Bigg[& \frac{-(n+1)\left(e^{-s/2}e^{i(n+1)(\phi-\theta)} - e^{-s}\right)}{\left(1 - e^{-s/2(n+1)}e^{i(\theta-\phi)}\right)\left(1 - e^{-s/2(n+1)}e^{i(\phi-\theta)}\right)} \\
& + \frac{\left|1 - e^{-s/2}e^{i(n+1)(\theta-\phi)}\right|^2 \left(e^{-s/2(n+1)}e^{i(\phi-\theta)} - e^{-s/(n+1)}\right)}{\left(1 - e^{-s/2(n+1)}e^{i(\theta-\phi)}\right)^2 \left(1 - e^{-s/2(n+1)}e^{i(\phi-\theta)}\right)^2} \Bigg] d\phi \\
\sim \sigma^2 e^{\beta n} \int_{-\pi}^{\pi} & f(\phi) \frac{c_n^1 \cdot (n+1)}{\left(1 - e^{-s/2(n+1)}e^{i(\theta-\phi)}\right)\left(1 - e^{-s/2(n+1)}e^{i(\phi-\theta)}\right)} d\phi \\
+ \sigma^2 e^{\beta n} \int_{-\pi}^{\pi} & \frac{c_n^2 f(\phi)\left(e^{-s/2(n+1)}e^{i(\phi-\theta)} - e^{-s/(n+1)}\right)}{\left(1 - e^{-s/2(n+1)}e^{i(\theta-\phi)}\right)^2 \left(1 - e^{-s/2(n+1)}e^{i(\phi-\theta)}\right)^2} d\phi \\
= B_1^1 + B_1^2. &
\end{aligned}$$

From our work on B_0 we know that

$$B_1^1 \sim c_n^1 \sigma^2 e^{\beta n}(n+1)^2 f(\theta)\frac{2\pi}{s}.$$

To handle B_1^2 we can apply a procedure similar to the one used on B_0^1 and B_0^2. We then have

$$\begin{aligned}
B_1^2 &\sim \sigma^2 e^{\beta n} \int_{\theta-(n+1)^{-\frac{1}{4}}}^{\theta+(n+1)^{-\frac{1}{4}}} f(\phi) \frac{c_n^2\left(e^{-s/2(n+1)}e^{i(\phi-\theta)} - e^{-s/(n+1)}\right)}{\left(1 - 2e^{-s/2(n+1)}\cos(\theta-\phi) + e^{-s/(n+1)}\right)^2} d\phi \\
&\sim 2f(\theta)\sigma^2 e^{\beta n} \int_{\theta}^{\theta+(n+1)^{-\frac{1}{4}}} \frac{c_n^2\left(\frac{s}{2(n+1)} - \frac{(\theta-\phi)^2}{2}\right)}{\left((\theta-\phi)^2 + \frac{s^2}{4(n+1)^2}\right)^2} d\phi \\
&= 2f(\theta)\sigma^2 e^{\beta n}\frac{c_n^2(n+1)^2}{s^2}\Bigg[\frac{-(4s(n+1)+s^2)(\theta-\phi)}{s^2+4(n+1)^2(\theta-\phi)^2} \\
&\quad + \left(\frac{s}{2(n+1)} - 2\right)\arctan\left(\frac{2}{s}(n+1)(\theta-\phi)\right)\Bigg]\Bigg|_{\theta}^{\theta+(n+1)^{-\frac{1}{4}}}
\end{aligned}$$

$$\sim 2\pi\sigma^2 e^{\beta n} f(\theta)(n+1)^2 \frac{c_n^2}{s^2}.$$

Using the fact that in the independent case

$$B_1(z) = \sigma^2 e^{\beta n} \frac{-(n+1)e^{-s}(1-e^{-s/(n+1)}) + e^{-s/(n+1)}(1-e^{-s})}{(1-e^{-s/(n+1)})^2},$$

we can again solve for the constants using the same procedure as before. Thus,

$$\sigma^2 e^{\beta n}(n+1)^2 \frac{c_n^1}{s} + \frac{c_n^2}{s^2} \sim \sigma^2 e^{\beta n} \frac{-(n+1)e^{-s}\left(1-e^{-s/(n+1)}\right) + e^{-s/(n+1)}\left(1-e^{-s}\right)}{\left(1-e^{-s/(n+1)}\right)^2},$$

from which it follows that

$$B_1(z) \sim 2\pi\sigma^2 e^{\beta n} \frac{-(n+1)e^{-s}\left(1-e^{-s/(n+1)}\right) + e^{-s/(n+1)}\left(1-e^{-s}\right)}{\left(1-e^{-s/(n+1)}\right)^2} f(\theta). \tag{4.3}$$

Lastly, since f is real-valued, it is easy to see that

$$B_1(z) \sim \overline{B_1(z)}$$

as well. Thus, plugging (4.2) and (4.3) into (4.1) gives us

$$\begin{aligned}
\mathrm{E}[\nu_n(D(r))] &= \frac{1}{2\pi}\int_0^{2\pi} F(re^{i\theta})d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{\overline{B_1(re^{i\theta})}}{B_0(re^{i\theta})}d\theta \\
&\sim \frac{1}{2\pi}\int_0^{2\pi} \left[\frac{-(n+1)e^{-s}}{1-e^{-s}} + \frac{e^{-s/(n+1)}}{1-e^{-s/(n+1)}}\right] d\theta \\
&= \frac{-(n+1)e^{-s}}{1-e^{-s}} + \frac{e^{-s/(n+1)}}{1-e^{-s/(n+1)}} \\
&\sim (n+1)\frac{1-e^{-s}(1+s)}{s(1-e^{-s})}.
\end{aligned}$$

Letting $s \to 0$, we have

$$\sim (n+1)\left(\frac{1}{2} - \frac{s}{3}\right),$$

as claimed. □

Theorem 4.2. *Let $r = e^{\beta/2 - 1/2(k+1)}$. Then,*

$$\lim_{n\to\infty} \mathrm{E}[\nu_n(D(r))] \sim k+1,$$

as $k \to \infty$.

Proof. From (3.7) we have

$$\mathrm{E}[\nu_n(D(r))] = \frac{1}{2\pi}\int_0^{2\pi} F(re^{i\theta})d\theta,$$

where

$$r = e^{\beta/2 - 1/2(k+1)}, \quad z = re^{i\theta}.$$

We will start by applying the Lebesgue dominated convergence theorem to the components of F, which results in the formula

$$\lim_{n\to\infty} F(z) = \frac{\overline{C(z)}}{B(z)},$$

where

$$(4.4)\quad \begin{aligned} B(z) &= \lim_{n\to\infty} e^{-\beta n} B_0(z) = \sigma^2 \int_{-\pi}^{\pi} f(\phi) \frac{1}{1 - ze^{-\beta/2}e^{-i\phi}} \cdot \frac{1}{1 - \overline{z}e^{-\beta/2}e^{i\phi}} d\phi, \\ C(z) &= \lim_{n\to\infty} e^{-\beta n} B_1(z) = \sigma^2 \int_{-\pi}^{\pi} f(\phi) \frac{1}{1 - ze^{-\beta/2}e^{-i\phi}} \cdot \frac{\overline{z}e^{-\beta/2}e^{i\phi}}{(1 - \overline{z}e^{-\beta/2}e^{i\phi})^2} d\phi. \end{aligned}$$

Applying the Lebesgue dominated convergence theorem once more,

$$(4.5)\quad \lim_{n\to\infty} \mathrm{E}[\nu_n(D(r))] = \frac{1}{2\pi} \int_0^{2\pi} \lim_{n\to\infty} F(re^{i\theta}) d\theta.$$

We can apply an analysis similar to the one used for $B_0(z)$ and $B_1(z)$ in the proof of Theorem 4.1. Then, for $B(z)$ we have

$$\begin{aligned} B(z) &= \sigma^2 \int_{-\pi}^{\pi} f(\phi) \frac{1}{\left(1 - e^{-1/2(k+1)}e^{i(\theta-\phi)}\right)\left(1 - e^{-1/2(k+1)}e^{i(\phi-\theta)}\right)} d\phi \\ &\sim \sigma^2 \int_{\theta-(k+1)^{-\frac{1}{4}}}^{\theta+(k+1)^{-\frac{1}{4}}} f(\phi) \frac{1}{1 - 2e^{-1/2(k+1)}\cos(\theta-\phi) + e^{-1/(k+1)}} d\phi \\ &\sim \sigma^2 f(\theta) \int_{\theta-(k+1)^{-\frac{1}{4}}}^{\theta+(k+1)^{-\frac{1}{4}}} \frac{1}{(\theta-\phi)^2 + \frac{1}{4(k+1)^2}} d\phi \\ &\sim 2\pi\sigma^2 f(\theta)(k+1). \end{aligned}$$

Similarly,

$$\begin{aligned} C(z) &= \sigma^2 \int_{-\pi}^{\pi} f(\phi) \frac{e^{-1/2(k+1)}e^{i(\phi-\theta)} - e^{-1/(k+1)}}{\left(1 - e^{-1/2(k+1)}e^{i(\theta-\phi)}\right)^2\left(1 - e^{-1/2(k+1)}e^{i(\phi-\theta)}\right)^2} d\phi \\ &\sim \sigma^2 \int_{\theta-(k+1)^{-\frac{1}{4}}}^{\theta+(k+1)^{-\frac{1}{4}}} f(\phi) \frac{e^{-1/2(k+1)}\cos(\phi-\theta) - e^{-1/(k+1)}}{\left(1 - 2e^{-1/2(k+1)}\cos(\theta-\phi) + e^{-1/(k+1)}\right)^2} d\phi \\ &\sim \sigma^2 \int_{\theta-(k+1)^{-\frac{1}{4}}}^{\theta+(k+1)^{-\frac{1}{4}}} \frac{\frac{1}{2(k+1)} - \frac{(\theta-\phi)^2}{2}}{\left((\theta-\phi)^2 + \frac{1}{4(k+1)^2}\right)^2} d\phi \\ &\sim 2\pi\sigma^2 f(\theta)(k+1)^2. \end{aligned}$$

Plugging into (4.5),

$$\begin{aligned} \lim_{n\to\infty} \mathrm{E}[\nu_n(D(r))] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \lim_{n\to\infty} F(re^{i\theta}) d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\overline{C(re^{i\theta})}}{B(re^{i\theta})} d\theta \\ &\sim \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{2\pi\sigma^2 f(\theta)(k+1)^2}{2\pi\sigma^2 f(\theta)(k+1)} d\theta \end{aligned}$$

$$\sim k + 1.$$

□

5. CONCLUSIONS

We began by mentioning a general result that gives an idea of the limiting behavior of the roots of a random polynomial that has dependent mean zero complex Gaussian coefficients, where the variance is exponentially increasing or decreasing. By then adding certain restrictions to the covariance function, we were able to derive more accurate results, which give more detailed information on the way in which this occurs. However, even then we were only able to do this by using approximations and asymptotic values. Without having more specific knowledge of the covariance function and the spectral density, we do not see a way to make these results more exact. On the other hand, if one were to know the exact expression of the spectral density it is likely that even more details on the specifics of the behavior could be obtained.

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