THE K-LEVEL CROSSINGS OF A RANDOM ALGEBRAIC POLYNOMIAL WITH DEPENDENT COEFFICIENTS

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Abstract. For a random polynomial with standard normal coefficients, two cases of the K-level crossings have been considered by Farahmand. For independent coefficients, Farahmand derived an asymptotic value for the expected number of level crossings, even if K grows to infinity. Alternatively, he showed that coefficients with a constant covariance have half as many crossings. Given these results, the purpose of this paper is to study the behavior for dependent standard normal coefficients where the covariance is decaying. Using similar techniques to Farahmand, we will show that for a wide range of covariance functions behavior similar to the independent case can be expected.

1. Introduction

For the random polynomial given by

$$P_n(x) = \sum_{k=0}^{n} X_k x^k, \tag{1.1}$$

consider the problem of computing the expected number of real zeros for the equation $P_n(x) = K$, where K is a given constant. These are known as the K-level crossings of $P_n(x)$. The random polynomial $P_n(x)$ is an example of a non-stationary random process. Random processes and their level crossings have applications in various fields, including the study of random noise [8, 9] and wireless communcations [1]. A further discussion of random processes and their applications can be found in [6].

In this paper we will focus on the K-level crossings of the random polynomial $P_n(x)$ when the coefficients are assumed to be standard normal random variables.

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For such coefficients, two separate cases were previously considered by [4, 5]. The first assumes the coefficients are independent. Here, Farahmand derived an asymptotic value for the expected number of level crossings, for both K bounded and K growing with n. The second case deals with dependent coefficients with a constant covariance ρ, where $\rho \in (0, 1)$. What Farahmand showed here was that the constant covariance causes the expected number of level crossings to be reduced by half.

Motivated by the results of Farahmand in these two extreme cases, it would be of interest to see what happens when there is some decay of the covariance. For the special case of $K = 0$, it was shown in [7] that behavior similar to the independent case can be expected when certain assumptions are made on the covariance function. With that in mind, we would like to see if this behavior holds more generally, for non-zero values of K. So, the goal of this paper is to further study the behavior of the crossings for the dependent case, when there is some decay of the covariance between the coefficients.

The setup for this problem will be as follows. Let $X_0, X_1, \ldots$ be a stationary sequence of normal random variables, where the covariance function is given by

$$\Gamma(k) = \mathrm{E}\left[X_0 X_k\right], \qquad \Gamma(0) = 1.$$

Similar to our investigation in [7], we will express $\Gamma(k)$ using the spectral density. That is,

$$\Gamma(k) = \int_{-\pi}^{\pi} e^{-ik\phi} f(\phi) d\phi, \tag{1.2}$$

where $f(\phi)$ is the spectral density of the covariance function (in addition to the discussion in [7], see [2] and [3] for further references). By imposing certain conditions on the spectral density, for the random polynomial $P_n(x)$ given by (1.1), we will be able to study the level crossings for a wide range of covariance functions.

Our work will cover two different assumptions on K, similar to those considered by Farahmand. As long as the spectral density has nice enough properties, similar behavior to the independent case can be expected. Assuming K is bounded, if we require that the spectral density is positive and in $C([-\pi,\pi])$, we will be able to show that the expected number of level crossings will behave asymptotically like $\frac{2}{\pi}\log n$ as $n\to\infty$. On the other hand, if K is allowed to grow along with n, such that $K=o\left(\sqrt{\frac{n}{\log\log n}}\right)$, and if the spectral density is positive and in $C^1([-\pi,\pi])$, the expected number of crossings in the interval $(-1,1)$ is reduced. These results will be proved using the techniques developed in [4, 5], as well as the spectral density of the covariance function. We will also make use of several results from [7], which in turn draws heavily from [10]. Letting $N_K(\alpha,\beta)$ be the number of K-level crossings of $P_n(x)$ in the interval (α,β), the main theorem is formulated as follows.

Theorem 1.1. *Assume that the spectral density exists and is strictly positive.*

(i) *For K bounded and $f(\phi)\in C([-\pi,\pi])$ we have*

$$\mathrm{E}\left[N_K(-1,1)\right]=\mathrm{E}\left[N_K(-\infty,-1)+N_K(1,\infty)\right]\sim\frac{1}{\pi}\log n.$$

(ii) *For $K=o\left(\sqrt{\frac{n}{\log\log n}}\right)$ and $f(\phi)\in C^1([-\pi,\pi])$ we have*

$$\mathrm{E}\left[N_K(-1,1)\right]=\frac{1}{\pi}\log\frac{n}{K^2}+O(\log\log n),$$
$$\mathrm{E}\left[N_K(-\infty,-1)+N_K(1,\infty)\right]=\frac{1}{\pi}\log n+O(\log\log n).$$

To begin with, using the Kac-Rice formula derived in [6], we have

$$\begin{aligned}\mathrm{E}\left[N_K(\alpha,\beta)\right]&=\frac{1}{\pi}\int_\alpha^\beta\frac{\sqrt{AC-B^2}}{A}\exp\left(-\frac{K^2C}{2(AC-B^2)}\right)dx\\&\quad+\frac{1}{\pi}\int_\alpha^\beta\frac{\sqrt{2}|BK|}{A^{3/2}}\exp\left(-\frac{K^2}{2A}\right)\operatorname{erf}\left(\frac{|-BK|}{\sqrt{2A(AC-B^2)}}\right)dx\\&=\int_\alpha^\beta F_1dx+\int_\alpha^\beta F_2dx,\end{aligned}\tag{1.3}$$

where $A(x) = \mathrm{E}[P_n^2(x)]$, $B(x) = \mathrm{E}[P_n(x)P_n'(x)]$, and $C(x) = \mathrm{E}[(P_n'(x))^2]$. From (2.6), (2.7), and (2.8) in [7], we have

(1.4)

$$\begin{aligned}
A &= \sum_{k=0}^{n}\sum_{j=0}^{n}\Gamma(k-j)x^{k+j} \\
&= \int_{-\pi}^{\pi}\frac{1-x^{n+1}e^{-i(n+1)\phi}}{1-xe^{-i\phi}}\cdot\frac{1-x^{n+1}e^{i(n+1)\phi}}{1-xe^{i\phi}}f(\phi)d\phi, \\
B &= \sum_{k=0}^{n}\sum_{j=0}^{n}\Gamma(k-j)kx^{k+j-1} \\
&= \int_{-\pi}^{\pi}\left(\frac{1-x^{n+1}e^{-i(n+1)\phi}}{1-xe^{-i\phi}}\right) \\
&\quad\cdot\left(\frac{-(n+1)x^n e^{i(n+1)\phi}(1-xe^{i\phi})-(1-x^{n+1}e^{i(n+1)\phi})(-e^{i\phi})}{(1-xe^{i\phi})^2}\right)f(\phi)d\phi, \\
C &= \sum_{k=0}^{n}\sum_{j=0}^{n}\Gamma(k-j)kjx^{k+j-2} \\
&= \int_{-\pi}^{\pi}\left(\frac{-(n+1)x^n e^{-i(n+1)\phi}(1-xe^{-i\phi})-(1-x^{n+1}e^{-i(n+1)\phi})(-e^{-i\phi})}{(1-xe^{-i\phi})^2}\right) \\
&\quad\cdot\left(\frac{-(n+1)x^n e^{i(n+1)\phi}(1-xe^{i\phi})-(1-x^{n+1}e^{i(n+1)\phi})(-e^{i\phi})}{(1-xe^{i\phi})^2}\right)f(\phi)d\phi.
\end{aligned}$$

2. Expected Number of Level Crossings on $(-1,1)$

To prove Theorem 1.1 we will start as in [4, 5]. That is, our first step will be to show that the contribution from the integral of F_2 on $(-1,1)$ is negligible.

Lemma 2.1. *For $f(\phi)$ continuous and positive we have*

$$\int_{-1}^{1} F_2 dx = o(\log\log n).$$

Proof. Since $f(\phi)$ is a continuous, positive function, we can find constants $c_1, c_2 > 0$ such that $\frac{c_1}{2\pi} > f(\phi) > \frac{c_2}{2\pi}$ for any $\phi \in [-\pi,\pi]$. Now, for the interval $(-1+\frac{\log\log n}{n}, 1-\frac{\log\log n}{n})$ we have

$$A \sim \int_{-\pi}^{\pi}\frac{1}{(1-xe^{-i\phi})(1-xe^{i\phi})}f(\phi)d\phi,$$

from which we can then derive the lower bound

$$A \geq \frac{c_2}{2\pi}\int_{-\pi}^{\pi}\frac{1}{(1-xe^{-i\phi})(1-xe^{i\phi})}d\phi = \frac{c_2}{1-x^2}. \tag{2.1}$$

Using the fact that $f \equiv \frac{1}{2\pi}$ in the independent case, we can derive an upper bound as well, where

$$\begin{aligned} A &\leq \frac{c_1}{2\pi}\int_{-\pi}^{\pi}\frac{\left(1-x^{n+1}e^{-i(n+1)\phi}\right)\left(1-x^{n+1}e^{i(n+1)\phi}\right)}{(1-xe^{-i\phi})(1-xe^{i\phi})}d\phi \\ &= c_1\frac{1-x^{2n+2}}{1-x^2} \leq \frac{c_1}{1-x^2}. \end{aligned} \tag{2.2}$$

Notice that this upper bound holds on the entire interval $(-1,1)$. Next, from the proof of Lemma 3.1 in [7] we know that

$$|B| \sim \int_{-\pi}^{\pi}\left|\frac{e^{i\phi}}{(1-xe^{-i\phi})(1-xe^{i\phi})^2}\right| f(\phi)d\phi,$$

which implies

$$|B| \leq \frac{1}{1-|x|}\int_{-\pi}^{\pi}\frac{1}{(1-xe^{-i\phi})(1-xe^{i\phi})}f(\phi)d\phi \sim \frac{1}{1-|x|}A.$$

By the above and (2.1) we have

$$\frac{|B|}{A^{3/2}} \leq \frac{1}{1-|x|}\left(\frac{1-x^2}{c_2}\right)^{1/2} \leq \sqrt{\frac{2}{c_2}}\frac{1}{(1-|x|)^{1/2}},$$

while from (2.2) we get

$$\exp\left(\frac{-K^2}{2A}\right) \leq \frac{1}{1+\frac{K^2(1-x^2)}{2c_1}} \leq \frac{1}{1+\frac{K^2(1-|x|)}{2c_1}}.$$

Since $\operatorname{erf}(x) \leq 1$, we then have

$$\int_{-1+\frac{\log\log n}{n}}^{1-\frac{\log\log n}{n}} F_2 dx \leq 2\sqrt{\frac{2}{c_2}}\int_0^{1-\frac{\log\log n}{n}}\frac{|K|(1-x)^{-1/2}}{1+\frac{K^2(1-x)}{2c_1}}dx = O(1). \tag{2.3}$$

Next, for $x \in (-1,-1+\frac{\log\log n}{n}) \cup (1-\frac{\log\log n}{n},1)$, we can use (1.4) and (2.2) to get

$$|B| \leq \frac{n}{|x|}\sum_{k=0}^{n}\sum_{j=0}^{n}\Gamma(k-j)|x|^{k+j} \leq \frac{nc_1}{|x|}\sum_{k=0}^{n}x^{2k}.$$

Also,

$$A \geq \frac{c_2}{2\pi} \int_{-\pi}^{\pi} \frac{\left(1 - x^{n+1} e^{-i(n+1)\phi}\right)\left(1 - x^{n+1} e^{i(n+1)\phi}\right)}{(1 - xe^{-i\phi})(1 - xe^{i\phi})} d\phi = c_2 \sum_{k=0}^{n} x^{2k},$$

from which it then follows that

$$\begin{aligned} \frac{|B|}{A^{3/2}} &\leq nc \left(\sum_{k=0}^{n} x^{2k} \right)^{-1/2} \\ &\leq nc \left(\sum_{k=0}^{n} \left(1 - \frac{\log\log n}{n} \right)^{2k} \right)^{-1/2} \sim c \left(n \log\log n \right)^{1/2}. \end{aligned}$$

Thus,

$$\frac{\sqrt{2}}{\pi} \int_{1 - \frac{\log\log n}{n}}^{1} F_2 \leq c \int_{1 - \frac{\log\log n}{n}}^{1} |K| \left(n \log\log n \right)^{1/2} = o\left(\log\log n \right).$$

We can use a similar procedure to show that

$$\frac{\sqrt{2}}{\pi} \int_{-1}^{-1 + \frac{\log\log n}{n}} F_2 = o(\log\log n),$$

which then proves the claim. □

We will next show that the expected number of crossings on the intervals $(0, 1 - \frac{1}{\log n})$, $(1 - \frac{\log\log n}{n}, 1)$, $(-1 + \frac{1}{\log n}, 0)$ and $(-1, -1 + \frac{\log\log n}{n})$ is negligible.

Lemma 2.2. *Assume $f(\phi)$ is continuous and positive. For the intervals $(-1, -1 + \frac{\log\log n}{n})$, $(-1 + \frac{1}{\log n}, 0)$, $(0, 1 - \frac{1}{\log n})$, and $(1 - \frac{\log\log n}{n}, 1)$, the expected number of crossings is $O(\log\log n)$.*

Proof. To start, we note that since the quantity $\frac{K^2 C}{AC - B^2}$ is never negative, the inequality

$$\exp\left(-\frac{K^2 C}{2\left(AC - B^2 \right)} \right) \leq 1$$

holds in general. It follows that

$$\int_{\alpha}^{\beta} F_1 dx \leq \frac{1}{\pi} \int_{\alpha}^{\beta} \frac{\sqrt{AC - B^2}}{A} dx. \tag{2.4}$$

Since this last formula is simply the expected number of zeros of $P_n(x)$ on the interval (α, β), applying Lemma 2.1 from above, along with Lemma 2.1 from [7], the result then follows. □

The last lemma of this section will derive asymptotic values for the integral of F_1 on the intervals $(-1 + \frac{\log\log n}{n}, -1 + \frac{1}{\log n})$ and $(1 - \frac{1}{\log n}, 1 - \frac{\log\log n}{n})$.

Lemma 2.3. *For the integral of* F_1 *on the intervals* $(-1 + \frac{\log\log n}{n}, -1 + \frac{1}{\log n})$ *and* $(1 - \frac{1}{\log n}, 1 - \frac{\log\log n}{n})$ *we have the following:*

(i) *For* K *bounded and* $f \in C([-\pi, \pi])$,

$$\frac{1}{\pi}\int_{-1+\frac{\log\log n}{n}}^{-1+\frac{1}{\log n}} F_1 = \frac{1}{\pi}\int_{1-\frac{1}{\log n}}^{1-\frac{\log\log n}{n}} F_1 \sim \frac{1}{2\pi}\log n.$$

(ii) *For* $K = o\left(\sqrt{\frac{n}{\log\log n}}\right)$ *and* $f \in C^1([-\pi, \pi])$,

$$\frac{1}{\pi}\int_{-1+\frac{\log\log n}{n}}^{-1+\frac{1}{\log n}} F_1 = \frac{1}{\pi}\int_{1-\frac{1}{\log n}}^{1-\frac{\log\log n}{n}} F_1 = \frac{1}{2\pi}\log\left(\frac{n}{K^2}\right) + O(\log\log n).$$

Proof. We will follow a similar procedure to that used by [4, 5]. That is, an asymptotic value for the integral of F_1 will be obtained by deriving upper and lower bounds for the integral, whereupon the true asymptotic value will then lie between these. Let $g(y) = y\frac{\log n}{\log\log n}$. Let $x = 1 - y \in (1 - \frac{1}{\log n}, 1 - \frac{\log\log n}{n})$ and assume $f(\phi) \in C([-\pi, \pi])$ and K is bounded. Using (3.2), (3.5), and (3.9) from [7], along with (1.3), we have

$$\frac{1}{\pi}\int_{1-\frac{1}{\log n}}^{1-\frac{\log\log n}{n}} F_1 \sim \frac{1}{\pi}\int_{\frac{\log\log n}{n}}^{\frac{1}{\log n}} \frac{1}{2y}\left(1 - \frac{K^2 y}{2f(0)\arctan\left(\frac{g(y)}{y}\right)}\right) dy \sim \frac{1}{2\pi}\log n. \tag{2.5}$$

Next, let $f(\phi) \in C^1([-\pi, \pi])$ and $K = o\left(\sqrt{\frac{n}{\log\log n}}\right)$. Using (3.2), (3.5), and (3.9) from [7] once more gives

$$\frac{K^2 C}{2(AC - B^2)} = \frac{K^2 y}{2f(0)\arctan\left(\frac{g(y)}{y}\right)} + O\left(\frac{K^2 y^2}{g(y)}\right).$$

Now, we can choose positive constants a_1 and a_2 such that for large enough n

$$\frac{a_1 K^2 y}{2f(0)\arctan\left(\frac{g(y)}{y}\right)} \le \frac{K^2 y}{2f(0)\arctan\left(\frac{g(y)}{y}\right)} + O\left(\frac{K^2 y^2}{g(y)}\right) \le \frac{a_2 K^2 y}{2f(0)\arctan\left(\frac{g(y)}{y}\right)},$$

which along with (3.11) from [7] then yields

$$\left[\frac{1}{2y} + O\left(\frac{1}{g(y)}\right)\right] \exp\left(\frac{-a_2 K^2 y}{2f(0)\arctan\left(\frac{g(y)}{y}\right)}\right) \le F_1 \le \left[\frac{1}{2y} + O\left(\frac{1}{g(y)}\right)\right] \exp\left(\frac{-a_1 K^2 y}{2f(0)\arctan\left(\frac{g(y)}{y}\right)}\right). \tag{2.6}$$

For $i = 1, 2$ we have

$$\left[\frac{1}{2y} + O\left(\frac{1}{g(y)}\right)\right] \exp\left(\frac{-a_i K^2 y}{2f(0)\arctan\left(\frac{g(y)}{y}\right)}\right) = \frac{1}{2y} \exp\left(\frac{-a_i K^2 y}{2f(0)\arctan\left(\frac{g(y)}{y}\right)}\right) + O\left(\frac{1}{g(y)}\right). \tag{2.7}$$

Thus, using an argument similar to the one on page 706 in [4],

(2.8)

$$\frac{1}{\pi}\int_{\frac{\log\log n}{n}}^{\frac{1}{\log n}}\left[\frac{1}{2y}\exp\left(\frac{-a_iK^2y}{2f(0)\arctan\left(\frac{g(y)}{y}\right)}\right)+O\left(\frac{1}{g(y)}\right)\right]dy$$
$$=\frac{1}{\pi}\int_{\frac{\log\log n}{n}}^{\frac{1}{\log n}}\frac{1}{2y}\exp\left(-cK^2y\right)dy+O\left(\log\log n\right)$$
$$\left(\text{where } c=a_i\left[2f(0)\arctan\left(\tfrac{g(y)}{y}\right)\right]^{-1}\right)$$
$$=\frac{1}{2\pi}\left[\log\left(cK^2\frac{1}{\log n}\right)-\log\left(cK^2\frac{\log\log n}{n}\right)\right]$$
$$+\frac{1}{2\pi}\int_0^{cK^2\frac{\log\log n}{n}}\frac{1-e^{-t}}{t}dt-\frac{1}{2\pi}\int_0^{cK^2\frac{1}{\log n}}\frac{1-e^{-t}}{t}dt$$
$$=\frac{1}{2\pi}\log n+\frac{1}{2\pi}\int_0^{cK^2\frac{\log\log n}{n}}\frac{1-e^{-t}}{t}dt-\frac{1}{2\pi}\int_0^{cK^2\frac{1}{\log n}}\frac{1-e^{-t}}{t}dt+O\left(\log\log n\right).$$

Since we are assuming that $K^2\frac{\log\log n}{n}\to 0$ as $n\to\infty$, the first integral is $o(1)$. For the second, we again use an argument drawn from page 706 in [4] to get

$$=-\frac{1}{2\pi}\int_1^{cK^2\frac{1}{\log n}}\frac{1-e^{-t}}{t}dt-\frac{1}{2\pi}\int_0^1\frac{1-e^{-t}}{t}dt$$

(2.9)
$$=-\frac{1}{2\pi}\int_1^{cK^2\frac{1}{\log n}}\frac{1}{t}dt+\frac{1}{2\pi}\int_1^{cK^2\frac{1}{\log n}}\frac{e^{-t}}{t}dt+O(1)$$
$$=-\frac{1}{2\pi}\log K^2+O\left(\log\log n\right).$$

By (2.6), (2.7), (2.8), and (2.9) it then follows that

(2.10)
$$\frac{1}{\pi}\int_{1-\frac{1}{\log n}}^{1-\frac{\log\log n}{n}}F_1=\frac{1}{2\pi}\log\left(\frac{n}{K^2}\right)+O\left(\log\log n\right).$$

We will next handle the interval $(-1+\frac{\log\log n}{n},-1+\frac{1}{\log n})$. The relevant equations for A, B, and C are given by (3.15), (3.16), and (3.19) in [7]. Noting that these equations differ from the positive case only by the constants multiplying the leading terms, we can apply the exact same procedure as before to show that

(2.11)
$$\frac{1}{\pi}\int_{1-\frac{1}{\log n}}^{1-\frac{\log\log n}{n}} F_1 \sim \frac{1}{\pi}\int_{\frac{\log\log n}{n}}^{\frac{1}{\log n}} \frac{1}{2y}\left(1-\frac{K^2 y}{2f(\pi)\arctan\left(\frac{g(y)}{y}\right)}\right)dy \sim \frac{1}{2\pi}\log n,$$

for $f(\phi)\in C\left([-\pi,\pi]\right)$ and K bounded, and

$$\frac{1}{\pi}\int_{-1+\frac{\log\log n}{n}}^{-1+\frac{1}{\log n}} F_1 = \frac{1}{2\pi}\log\left(\frac{n}{K^2}\right)+O\left(\log\log n\right), \tag{2.12}$$

for $f(\phi)\in C^1\left([-\pi,\pi]\right)$ and $K = o\left(\sqrt{\frac{n}{\frac{\log\log n}{n}}}\right)$. Combined with (2.5) and (2.10), this completes the proof. □

3. Expected Number of Level Crossings on $(-\infty,-1)$ and $(1,\infty)$

Now that we have derived the expected number of zeros for $(-1,1)$, this last section will consider the remaining intervals $(-\infty,-1)$ and $(1,\infty)$. We will start with the latter. As done by [4, 5], let $x=\frac{1}{z}$. Using (1.4), for $z\in(0,1)$ we have

$$\begin{aligned}
A\left(\frac{1}{z}\right) &= z^{-2n}\int_{-\pi}^{\pi}\frac{1-z^{n+1}e^{i(n+1)\phi}}{1-ze^{i\phi}}\cdot\frac{1-z^{n+1}e^{-i(n+1)\phi}}{1-ze^{-i\phi}}f(\phi)d\phi,\\
B\left(\frac{1}{z}\right) &= -z^{-2n+1}\int_{-\pi}^{\pi}\frac{1-z^{n+1}e^{i(n+1)\phi}}{1-ze^{i\phi}}\\
&\quad\cdot\frac{-(n+1)\left(1-ze^{-i\phi}\right)+1-z^{n+1}e^{-i(n+1)\phi}}{\left(1-ze^{-i\phi}\right)^2}f(\phi)d\phi,\\
C\left(\frac{1}{z}\right) &= z^{-2n+2}\int_{-\pi}^{\pi}\frac{-(n+1)\left(1-ze^{i\phi}\right)+1-z^{n+1}e^{i(n+1)\phi}}{\left(1-ze^{i\phi}\right)^2}\\
&\quad\cdot\frac{-(n+1)\left(1-ze^{-i\phi}\right)+1-z^{n+1}e^{-i(n+1)\phi}}{\left(1-ze^{-i\phi}\right)^2}f(\phi)d\phi.
\end{aligned} \tag{3.1}$$

As before, the first step is to get a bound for the integral of F_2.

Lemma 3.1.
$$\int_1^{\infty}F_2dx=\int_{-\infty}^{-1}F_2dx=o(1).$$

Proof. We have

$$\int_1^\infty F_2 dx \le \frac{\sqrt{2}}{\pi}\int_1^\infty \frac{|B(x)K|}{A^{3/2}(x)}dx = \frac{\sqrt{2}}{\pi}\int_0^1 \frac{1}{z^2}\frac{\left|B\left(\frac{1}{z}\right)K\right|}{A^{3/2}\left(\frac{1}{z}\right)}dz. \tag{3.2}$$

Let c_1 and c_2 be as in the proof of Lemma 2.1. Then, for $z \in (-1,0)\cup(0,1)$,

$$\begin{aligned}\left|B\left(\frac{1}{z}\right)\right| &\le n|z|^{-2n+1}\sum_{k=0}^{n}\sum_{j=0}^{n}\Gamma(k-j)|z|^{2n-k-j}\\ &= n|z|^{-2n+1}A(|z|)\\ &\le c_1 n|z|^{-2n+1}\frac{1-z^{2n+2}}{1-z^2},\end{aligned}$$

where the last line is given by (2.2). Also,

$$\begin{aligned}A\left(\frac{1}{z}\right) &\ge z^{-2n}\frac{c_2}{2\pi}\int_{-\pi}^{\pi}\frac{1-z^{n+1}e^{i(n+1)\phi}}{(1-ze^{i\phi})}\cdot\frac{1-z^{n+1}e^{-i(n+1)\phi}}{(1-ze^{-i\phi})}d\phi\\ &= c_2 z^{-2n}\frac{1-z^{2n+2}}{1-z^2}.\end{aligned}$$

Thus,

$$\frac{\left|B\left(\frac{1}{z}\right)\right|}{A^{3/2}\left(\frac{1}{z}\right)} \le cn|z|^{n+1}\sqrt{\frac{1-z^2}{1-z^{2n+2}}}.$$

Consider the interval $(0, 1-\frac{1}{\sqrt{n}})$. Recalling that $K = o\left(\sqrt{\frac{n}{\log\log n}}\right)$, the above inequality yields

$$\begin{aligned}\frac{\sqrt{2}}{\pi}\int_0^{1-\frac{1}{\sqrt{n}}}\frac{1}{z^2}\frac{\left|B\left(\frac{1}{z}\right)K\right|}{A^{3/2}\left(\frac{1}{z}\right)}dz &\le c|K|\int_0^{1-\frac{1}{\sqrt{n}}} nz^{n-1}\sqrt{\frac{1-z^2}{1-z^{2n+2}}}\\ &\le c|K|\left(1-\frac{1}{\sqrt{n}}\right)^n = o(1).\end{aligned}$$

Next, for $z \in (1 - \frac{1}{\sqrt{n}}, 1)$ we have

$$\begin{aligned}
&\frac{\sqrt{2}}{\pi} \int_{1-\frac{1}{\sqrt{n}}}^{1} \frac{1}{z^2} \frac{\left|B\left(\frac{1}{z}\right)K\right|}{A^{3/2}\left(\frac{1}{z}\right)} dz \\
&\leq c|K| \int_{1-\frac{1}{\sqrt{n}}}^{1} n z^{n-1} \sqrt{\frac{1-z^2}{1-z^{2n+2}}} \\
&= c|K| z^n \sqrt{\frac{1-z^2}{1-z^{2n+2}}}\Bigg|_{1-\frac{1}{\sqrt{n}}}^{1} - c|K| \int_{1-\frac{1}{\sqrt{n}}}^{1} z^n \frac{d}{dz}\left(\sqrt{\frac{1-z^2}{1-z^{2n+2}}}\right) dz = o(1),
\end{aligned}$$

where the last line follows from the fact that

$$\frac{d}{dz}\left(\sqrt{\frac{1-z^2}{1-z^{2n+2}}}\right) = O\left(\sqrt{n}\right)$$

on $z \in (1 - \frac{1}{\sqrt{n}}, 1)$. Applying (3.2), this proves the result for $(1, \infty)$. Noting that the same argument works for $-z$, the result then follows for $(-\infty, -1)$ as well. □

The next lemma will evaluate the integral of F_1.

Lemma 3.2. (i) *For $f \in C([-\pi, \pi])$,*

$$\int_1^{\infty} F_1 dx = \int_{-\infty}^{-1} F_1 dx \sim \frac{1}{2\pi} \log n.$$

(ii) *For $f \in C^1([-\pi, \pi])$,*

$$\int_1^{\infty} F_1 dx = \int_{-\infty}^{-1} F_1 dx = \frac{1}{2\pi} \log n + O(\log \log n).$$

Proof. We will prove the result assuming that $f \in C^1([-\pi, \pi])$; the resulting argument will require only a few minor changes to prove the claim for $f \in C([-\pi, \pi])$. As in Lemma 2.3, this will be done by bounding the true asymptotic value between an upper and a lower bound. To start, we have the inequality

$$\int_1^{\infty} F_1 dx \leq \frac{1}{\pi} \int_1^{\infty} \frac{\sqrt{A(x)C(x) - B^2(x)}}{A(x)} dx.$$

Notice that the expression on the right is simply the expected number of real zeros of $P_n(x)$ on $(1,\infty)$. Similarly,

$$\int_{-\infty}^{-1} F_1 dx \le \frac{1}{\pi}\int_{-\infty}^{-1} \frac{\sqrt{A(x)C(x)-B^2(x)}}{A(x)}dx,$$

where now the expression on the right is the expected number of real zeros of $P_n(x)$ on $(-\infty,-1)$. Thus, Theorem 1.1 in [7] yields the upper bounds

$$\begin{aligned}\int_{1}^{\infty} F_1 dx &\le \frac{1}{2\pi}\log n + O\left(\log\log n\right),\\ \int_{-\infty}^{-1} F_1 dx &\le \frac{1}{2\pi}\log n + O\left(\log\log n\right).\end{aligned} \tag{3.3}$$

The rest of the proof will be devoted to the derivation of a lower bound.

Consider the interval $(1-\frac{1}{\log n}, 1-\frac{\log\log n}{n})$. Let $z = 1-y$, and recall that $g(y) = y\frac{\log n}{\log\log n}$. We will next need to make use of the asymptotic formulas

$$\begin{aligned}\int_{-\pi}^{\pi}\frac{f(\phi)d\phi}{\left(1-ze^{i\phi}\right)\left(1-ze^{-i\phi}\right)} &= \frac{2f(0)}{y}\arctan\left(\frac{g(y)}{y}\right) + O\left(\frac{1}{g(y)}\right),\\ \int_{-\pi}^{\pi}\frac{f(\phi)d\phi}{\left(1-ze^{i\phi}\right)\left(1-ze^{-i\phi}\right)^2} &= \frac{f(0)}{y^2}\arctan\left(\frac{g(y)}{y}\right) + O\left(\frac{1}{yg(y)}\right),\\ \int_{-\pi}^{\pi}\frac{f(\phi)d\phi}{\left(1-ze^{i\phi}\right)^2\left(1-ze^{-i\phi}\right)^2} &= \frac{f(0)}{y^3}\arctan\left(\frac{g(y)}{y}\right) + O\left(\frac{1}{y^2g(y)}\right),\end{aligned} \tag{3.4}$$

which are derived in the proof of Lemma 3.1 in [7]. Combining (3.1), (3.4), and some tedious algebra, we obtain the expression

$$\begin{aligned}&A\left(\frac{1}{z}\right)C\left(\frac{1}{z}\right)-B^2\left(\frac{1}{z}\right)\\ &= z^{-4n+2}\Bigg[\int_{-\pi}^{\pi}\frac{f(\phi)d\phi}{\left(1-ze^{i\phi}\right)\left(1-ze^{-i\phi}\right)}\cdot\int_{-\pi}^{\pi}\frac{f(\phi)d\phi}{\left(1-ze^{i\phi}\right)^2\left(1-ze^{-i\phi}\right)^2}\\ &\quad-\left(\int_{-\pi}^{\pi}\frac{f(\phi)d\phi}{\left(1-ze^{i\phi}\right)\left(1-ze^{-i\phi}\right)^2}\right)^2\\ &\quad+O\left((n+1)z^{n+1}\int_{-\pi}^{\pi}\frac{f(\phi)d\phi}{\left(1-ze^{i\phi}\right)\left(1-ze^{-i\phi}\right)}\cdot\int_{-\pi}^{\pi}\frac{f(\phi)d\phi}{\left(1-ze^{i\phi}\right)\left(1-ze^{-i\phi}\right)^2}\right)\Bigg]\end{aligned}$$

$$= (1-y)^{-4n+2}\left[\frac{f^2(0)}{y^4}\arctan^2\left(\frac{g(y)}{y}\right)+O\left(\frac{1}{y^3g(y)}\right)\right].$$

Thus,

$$\frac{\sqrt{A\left(\frac{1}{z}\right)C\left(\frac{1}{z}\right)-B^2\left(\frac{1}{z}\right)}}{A\left(\frac{1}{z}\right)} = (1-y)\left[\frac{1}{2y}+O\left(\frac{1}{g(y)}\right)\right]. \tag{3.5}$$

Also, if we refer to (3.4) once more,

$$C\left(\frac{1}{z}\right) \sim (1-y)^{-2n+2}\frac{2(n+1)^2f(0)}{y}\arctan\left(\frac{g(y)}{y}\right). \tag{3.6}$$

Applying (1.3) we then have

$$\begin{aligned}
&\int_1^\infty F_1\,dx = \\
&= \frac{1}{\pi}\int_0^1 \frac{1}{z^2}\frac{\sqrt{A\left(\frac{1}{z}\right)C\left(\frac{1}{z}\right)-B^2\left(\frac{1}{z}\right)}}{A\left(\frac{1}{z}\right)}\exp\left(-\frac{K^2C\left(\frac{1}{z}\right)}{2\left(A\left(\frac{1}{z}\right)C\left(\frac{1}{z}\right)-B^2\left(\frac{1}{z}\right)\right)}\right)dz \\
&\geq \frac{1}{\pi}\int_{1-\frac{1}{\log n}}^{1-\frac{\log\log n}{n}} \frac{1}{z^2}\frac{\sqrt{A(\frac{1}{z})C(\frac{1}{z})-B^2(\frac{1}{z})}}{A(\frac{1}{z})}\exp\left(-\frac{K^2C(\frac{1}{z})}{2\left(A(\frac{1}{z})C(\frac{1}{z})-B^2(\frac{1}{z})\right)}\right)dz \\
&= \frac{1}{\pi}\int_{\frac{\log\log n}{n}}^{\frac{1}{\log n}}\left[\frac{1}{2y(1-y)}\left[1+O\left(K^2(n+1)^2(1-y)^{2n}y^3\right)\right]+O\left(\frac{1}{g(y)}\right)\right]dy \\
&= \frac{1}{2\pi}\log n + O\left(\log\log n\right).
\end{aligned}$$

Noting that almost the exact same argument holds for $-z$,

$$\int_{-\infty}^{-1} F_1\,dx \geq \frac{1}{2\pi}\log n + O\left(\log\log n\right),$$

as well. Combined with (3.3), the claim then follows. □

Proof of Theorem 1.1. Combining the results of Lemmas 2.1, 2.2, 2.3, 3.1, and 3.2, Theorem 1.1 now follows. □

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References

[1] Abdi, A., Kaveh, M., 2002. Level crossing rate in terms of the characteristic function: A new approach for calculating the fading rate in diversity systems. IEEE Trans. Commun. 50, 1397-1400.

[2] Breiman, L., 1968. Probability. Addison-Wesley Publishing Company, Reading, Mass.

[3] Cramér, H., Leadbetter, M.R., 1967. Stationary and Related Stochastic Processes. John Wiley and Sons Inc., New York.

[4] Farahmand, K., 1986a. On the average number of real roots of a random algebraic equation. Ann. Probab. 14, 702-709.

[5] Farahmand, K., 1986b. On the average number of real roots of a random algebraic equation with dependent coefficients. J. Indian Math. Soc. 50, 49-58.

[6] Farahmand, K., 1998. Topics in Random Polynomials. Addison Wesley Longman, London.

[7] Matayoshi, J., in press. The real zeros of a random polynomial with dependent coefficients. Rocky Mt. J. Math. arXiv:0906.1996v2.

[8] Rice, S.O., 1945. Mathematical analysis of random noise. Reprinted in: Selected Papers on Noise and Stochastic Processes, N. Wax (ed.), Dover, New York, 1954, 133-294.

[9] Rice, S.O., 1958. Distribution of the duration of fades in radio transmission: Gaussian noise model. Bell. System Tech. J. 37, 581-635.

[10] Sambandham, M., 1977. On a random algebraic equation. J. Indian Math. Soc. 41, 83-97.

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