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On the Zeros of Random Polynomials

Dissertation by

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The dissertation of Jeffrey Seiichi Matayoshi
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Dedication

To my parents, Brian and Marylyn.

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Acknowledgments

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Abstract of the Dissertation

On the Zeros of Random Polynomials

By

Jeffrey Seiichi Matayoshi

Doctor of Philosophy in Mathematics
University of California, Irvine, 2009
Professor Michael Cranston, Chair

In this dissertation we will prove several results pertaining to the properties of zeros of random polynomials. We will begin with a discussion of the expected number of real zeros for random polynomials with dependent standard normal coefficients. With certain restrictions imposed on the spectral density of the coefficients' covariance function, we will show that similar behavior to the independent case can be expected. Specifically, the value of the expected number of real zeros grows asymptotically like $\frac{2}{\pi}\log n$, as $n \to \infty$. After studying the real zeros, we will next consider the number of K-level crossings. Again imposing certain restrictions on the spectral density, an asymptotic value will be derived for the expected number of K-level crossings of random polynomials with dependent standard normal coefficients.

The next problem that will be considered is a study of the distribution of the complex zeros. Once more imposing restrictions on the spectral density, we will show that the complex zeros of random polynomials with dependent standard normal coefficients converge to the unit circle. Additionally, we will derive expressions approximating how fast this convergence happens. By then adapting the techniques used in the aforementioned problem, we will study the behavior of random polynomials which have applications to the GSM (Global System for Mobile Communications)/EDGE (Enhanced Data Rates for GSM Evolution) standard for mobile phones.

The last part of our work will consider the zeros of random sums of orthogonal

polynomials. For a random sum of the Chebyshev polynomials of the first kind, orthogonalized over the interval $[-1,1]$, we will show that the distribution of zeros converges to the corresponding equilibrium measure for this set. This result will lay the foundation for some further work in the area of random sums of orthogonal polynomials.

Introduction

A random polynomial is any polynomial of the form

$$P_n(x,\omega) = P_n(x) = \sum_{k=0}^{n} X_k x^k, \tag{1}$$

where the coefficients, X_k, are random variables, either independent or dependent, with some given distributions. The study of random polynomials is a broad area, with work being done on all the usual topics of interests for polynomials. This includes studying the properties of the zeros, maximums, minimums, and level crossings. It is an area with a rich history that contains results from such notable names as Littlewood [18], Kac [16], and Erdös [10]. In addition to being mathematically interesting in themselves, random polynomials have also found applications in various scientific fields. In quantum chaos, for example, it has been shown that random polynomials are good approximations for the eigenfunctions of the quantum Hamiltonian (see [32]; see also [3, 4] for further applications to quantum chaos). Another area of application would be in wireless communications [26], an example of which will be covered in more detail later. For a thorough discussion of the various properties of random polynomials, see the books by Bharucha-Reid and Sambandham [2], and Farahmand [13].

This dissertation will focus on studying the various properties of random polynomial zeros. In the first chapter, we will begin by giving a broad survey of the historical results in this area, including studies done on the expected number of real zeros, the expected number of level crossings, and various results on the distribution of the complex zeros. Additionally, we will also mention a result pertaining to the zeros of random orthogonal polynomials, as well as introduce some of the important concepts needed for our later work. The remaining four chapters will then be devoted to original research studying the aforementioned properties of random polynomial zeros. Chapter 2 discusses the expected number of real zeros of random polynomials with dependent coefficients, while Chapter 3 will discuss the expected number of K-level crossings. Chapter 4 will focus on the complex zeros of random polynomials, again with dependent coefficients, while also discussing an application of random polynomials to wireless communications. Lastly, Chapter 5 will contain a result on the distribution of the complex zeros of random polynomials that are composed of sums of orthogonal polynomials.

Chapter 1

Historical Results and Background

This chapter will be divided into several sections as follows. After starting with some preliminary definitions and notation, we will discuss several results on the expected number of real zeros for various types of random polynomials. This will include several results for independent coefficients, as well as the two main historical results for dependent coefficients. From here, we will move on to a couple of results on K-level crossings, followed by several important works on the distribution of the complex zeros. Next, we will discuss the work of Shiffman and Zelditch [29] on random sums of orthogonal polynomials, which will be fundamental to our result in Chapter 5. Finally, our last section will introduce several of the tools and concepts that will be needed in subsequent chapters.

1.1 Definitions/Notation

Throughout this work, we will assume that our random polynomial, $P_n(x)$, has the form given in (1). To emphasize the differences in the situations, we will refer to our polynomial as $P_n(x)$ in general or when focusing on the real zeros, while using $P_n(z)$ when referring specifically to the complex zeros. Also, for any given set $\Omega \in \mathbb{C}$, we will define $\nu_n(\Omega)$ to be the number of zeros of $P_n(x)$ in the set Ω. When dealing with zeros on the real line, we will use the more standard notation $N(\alpha, \beta)$ to represent the zeros of $P_n(x)$ on the interval (α, β). Similarly, $N_K(\alpha, \beta)$ will represent the number of K-level crossings on the interval (α, β).

1.2 Real Zeros

1.2.1 Independent Coefficients

One of the earliest results in the study of zeros of random polynomials is due to J.E. Littlewood and A.C. Offord in 1938 [18]. They considered three main cases of independent coefficients: those with a uniform distribution on $(-1, 1)$, a standard normal distribution, or a distribution taking the values of 1 and -1 with equal probability. While no explicit values for the expected number of real zeros were given, some upper bounds were obtained. It was shown that in the three main cases mentioned above, the expected number of real zeros is at most $25(\log n)^2 + 12\log n$. This is an interesting result because of the low value of the upper bound. Since a polynomial of degree n has at most n real zeros, it is slightly surprising that the expected number of real zeros can be bounded above by such a low asymptotic value.

The next major result (at least for our purposes) came from Mark Kac in 1943 [16]. Under the assumption that the coefficients are independent standard normals, Kac showed that the expected number of real zeros is on the order of $\frac{2}{\pi}\log n$, as $n \to \infty$. This result is noteworthy not only because it is one of the first (if not the first) explicit calculations of a random polynomial's expected number of real zeros, but also because it makes use of the Kac-Rice formula, which is a formula for computing a random polynomial's expected number of real zeros, in its earliest form. Using this formula, Kac proved that the expected number of real zeros on any interval (α, β) is given by

$$\text{(1.1)} \qquad \mathrm{E}\left[N(\alpha,\beta)\right] = \frac{1}{\pi}\int_{\alpha}^{\beta} \frac{(x^{4n} - n^2x^{2(n+1)} + 2(n^2-1)x^{2n} - n^2x^{2(n-1)} + 1)^{1/2}}{(x^2-1)^2(1+x^2+x^4+\ldots+x^{2n-2})}dx,$$

from which the above estimate is derived.

Now, from the formula above, it can be seen that the zeros tend to accumulate around -1 and 1. Before moving on, it may be useful to give some intuition into this behavior of the zeros. To see one possible reason for this, consider the polynomial

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n, \qquad |a_i| = 1.$$

If $x \in [-\frac{1}{2}, \frac{1}{2}]$, the a_0 term will dominate, and there will be no zeros. Similarly, if $x \in [2, \infty)$ or $x \in (-\infty, -2]$, the $a_n x^n$ term will dominate, and there will also be no zeros. So, for this example, the only possible zeros are in the intervals $(\frac{1}{2}, 2)$ and $(-2, -\frac{1}{2})$, which is somewhat similar to the behavior exhibited by the polynomials Kac studied. Additionally, since the coefficients of the polynomials studied by Kac are symmetrically distributed with mean zero, values of x near 1 and -1 would make it easiest to have cancellation between the terms, hence producing zeros. Thus, given these details, the behavior of the zeros is not as surprising as it might appear at first glance.

At this point the contributions of Rice should be mentioned. About the same time that Kac was working on the previously mentioned problem, S.O. Rice was working on problems related to stochastic noise. In the course of this work he independently developed a formula similar to Kac's for computing the expected number of real zeros of a random polynomial [20]. For this reason, the formula is credited to both of them and is now known as the "Kac-Rice" formula.

In 1955 Erdös and Offord [10] studied a class of random polynomials where the coefficients are either 1 or -1 with equal probability. Their result showed that the number of zeros in this case is also on the order of $\frac{2}{\pi} \log n$, for most of the equations. That is, the estimate holds except for on a small proportion of the total set of equations. This paper is important for our purposes because it develops a number of techniques which are later used by Sambandham. Sambandham, whose work will be discussed in the next section, employed these techniques in his study of random polynomials with dependent standard normal coefficients, where the covariance function is exponentially decaying.

Another result worth mentioning came from Edelman and Kostlan in 1995 [9]. They considered, like Kac, random polynomials with independent standard normal coefficients. They computed the expected number of real zeros and, again like Kac, derived the same estimate. However, Edelman and Kostlan used a geometric approach which was original and very different from any previous method. They considered the joint density function for a sequence of n standard normal random variables, which has the form $\frac{1}{(2\pi)^{n/2}}e^{-\frac{1}{2}\mathbf{v}\cdot\mathbf{v}}$, where $\mathbf{v} \in \mathbb{R}^{n+1}$. In the case of independent standard normals, the density becomes a function of the radius alone. If the radius is restricted to be 1, the values of this sequence of random variables are uniformly distributed on the unit sphere in $\mathbb{R}^{n+1}$. Defining the curve $\mathbf{x}(t) = [1, t, t^2, \ldots, t^n]^T$, it follows that $\forall \mathbf{v} \in \mathbb{R}^{n+1}$, $\mathbf{v} \cdot \mathbf{x}(t)$ is a realization of the random polynomial $P_n(t)$. Furthermore, this polynomial will have a zero when $\mathbf{v} \perp \mathbf{x}(t)$, for some $t \in \mathbb{R}$. By calculating the area on the sphere of the points which are orthogonal to $\mathbf{x}(t)$ for some t, and then dividing by the total area of the sphere, Edelman and Kostlan were able to derive Kac's formula for the expected number of zeros.

1.2.2 Dependent Coefficients

Once estimates are obtained for the independent standard normal case, an interesting next step is to analyze the behavior of the zeros when some dependence is assumed among the coefficients. The main work done for dependent standard normals can be divided into two cases. The first assumes that the coefficients have a constant covariance, ρ, where $\rho \in (0, 1)$. Under these assumptions, it has been shown that the expected number of real zeros is on the order of $\frac{1}{\pi} \log n$, or half of that in the independent case. Sambandham first studied this situation in 1976 [23], but a later result of his own contradicted his initial findings [25]. Later work by Miroshin [19] confirmed the validity of Sambandham's second result, which gave the correct estimate above.

The second case assumes the coefficients have an exponentially decaying covariance, which was considered by Sambandham in 1977 [24]. That is, letting $X_0, X_1, \ldots$ be a stationary sequence of standard normal random variables, the covariance function is given by

$$\mathrm{E}\left[X_i X_j\right] = \rho^{|i-j|}.$$

When $0 < \rho < 1/2$, Sambandham showed that the expected number of real zeros is on the order of $\frac{2}{\pi} \log n$. This result matches the value obtained in the independent case.

It is interesting to note some of the possible explanations that appear in the literature for these differing expected values. The first noticeable result is that the same estimate holds for the expected number of zeros in the independent case and the dependent case where the coefficients are exponentially correlated. A reasonable explanation for this is that the correlation between the coefficients is dying out at a fast enough rate (exponentially fast) to make them behave as if they are independent, resulting in a similar amount of zeros. On the other hand, there are half as many expected zeros in the dependent case with constant positive correlation. In this situation, because of the constant correlation, it is reasonable to expect that most of the coefficients would be of the same sign. If a polynomial has coefficients of all the same sign, the only possible zeros would be on the negative real line. This makes the expected number of zeros on the positive real line negligible, resulting in half the number of expected zeros as in the previous two cases.

A further direction of research motivated by these ideas is to try and identify at what point this result changes; that is, can we identify some critical rate of decay for the correlation function at which the expectation changes. We know if the covariance decays fast enough, the coefficients exhibit independent behavior. If the covariance has no decay, we have half as many zeros. Between these extremes it would be interesting to see what other types of behavior happens, and what properties of the covariance functions lead to this behavior. These questions are what motivated the work in Chapter 2.

One additional simple, yet interesting, case is when the coefficients have a constant covariance of 1. By using the Kac-Rice formula, it can be shown that the expected number of zeros is on the order of a constant. However, by performing the simple calculation

$$\mathrm{E}\left[(X_i - X_j)^2\right] = \mathrm{E}\left[X_i^2\right] + \mathrm{E}\left[X_j^2\right] - 2\mathrm{E}\left[X_i X_j\right] = 2 - 2 = 0, \qquad i \neq j,$$

it follows that $X_i = X_j$ a.e. for any i, j. Thus, the only possible zero of this polynomial would be -1, and only when n is odd.

1.3 K-Level Crossings

Consider the problem of computing the expected number of real zeros of the equation $P_n(x) = K$, where K is a given constant. These zeros are otherwise known as the K-level crossings of $P_n(x)$. When studying such crossings, two main situations are usually considered. The first assumes K is bounded, while in the second K is allowed to grow along with n. Two important works in this area are due to Farahmand.

For independent standard normal coefficients and K bounded, Farahmand [11] showed that the total expected number of K-level crossings on $(-\infty, \infty)$ is on the order of $\frac{2}{\pi} \log n$, which is the same result as for the real zeros. However, a curious behavior is exhibited when K is allowed to grow with n. Assuming $K = o(\sqrt{n})$, the expected number of crossings on the interval $(-1, 1)$ is reduced. Specifically, Farahmand showed that

$$\mathrm{E}\left[N_K(-1,0)\right] = \mathrm{E}\left[N_K(0,1)\right] \sim \frac{1}{2\pi} \log n, \quad \text{for } K \text{ bounded},$$
$$\mathrm{E}\left[N_K(-1,0)\right] = \mathrm{E}\left[N_K(0,1)\right] \sim \frac{1}{2\pi} \log\left(\frac{n}{K^2}\right), \quad \text{for } K = o\left(\sqrt{n}\right),$$

and

$$\mathrm{E}\left[N_K(-\infty,-1)\right] = \mathrm{E}\left[N_K(1,\infty)\right] \sim \frac{1}{2\pi} \log n.$$

In a later paper Farahmand [12] studied the K-level crossings for $P_n(x)$ when the coefficients are dependent standard normals, with a constant covariance $\rho \in (0, 1)$. Here, just as in the case of the real zeros, the number of level crossings is cut in half. The constant covariance causes the number of real crossings to be reduced drastically. Farahmand's result is given as

$$\mathrm{E}\left[N_K(-1,1)\right] \sim \frac{1}{2\pi} \log n, \quad \text{for } K \text{ bounded},$$
$$\mathrm{E}\left[N_K(-1,1)\right] \sim \frac{1}{2\pi} \log\left(\frac{n}{K^2}\right), \quad \text{for } K = o\left(\sqrt{\frac{n}{\log n}}\right),$$
$$\mathrm{E}\left[N_K(-\infty,-1)\right] + \mathrm{E}\left[N_K(1,\infty)\right] \sim \frac{1}{2\pi} \log n.$$

As in the independent case, there are also less crossings on $(-1, 1)$ when K is allowed to grow with n.

1.4 Complex Zeros

The first result in this area worth mentioning is due to Hammersley in 1956 [14]. In his classic paper, Hammersley studied the distribution of zeros for random polynomials with either real or complex Gaussian coefficients. He became interested in the problem when asked a question concerning the growth rates of an insect population. His task was to evaluate the zeros of polynomials whose coefficients were determined by experimental data. While, in his own words, he was not able to fully solve the problem given to him, he was able to derive explicit density functions for the distributions of the zeros. One fundamental result that can be shown by applying his formulas is that the zeros of random polynomials with independent complex Gaussian coefficients, having mean zero and variance one, tend to concentrate on the unit circle in the complex plane. Additionally, in his work Hammersley also succeeded in producing a more detailed version of Kac's result.

In 1995 Shepp and Vanderbei [28] studied more closely the complex zeros of random polynomials with independent standard normal coefficients. Using the argument principle, they derived a formula for computing the expected number of zeros in any measurable subset of the complex plane. Applying this formula, they then were able to prove several results about the distribution of the complex zeros. This included showing that the zeros tend to accumulate around the unit circle in the limit, and that they do so uniformly in the angle. Our work in Chapter 4 will be based on their techniques.

The last result on complex zeros that we will mention comes from Hughes and Nikeghbali in 1998 [15]. In their work, Hughes and Nikeghbali were able to show that under very general assumptions on the coefficients of $P_n(z)$, the same behavior observed by Shepp and Vanderbei will hold. That is, the zeros will accumulate around the unit circle in the limit, and they will do so uniformly in the angle. As a specific case, their result holds for Gaussian coefficients, with no restrictions on the dependence among the coefficients.

1.5 Random Sums of Orthogonal Polynomials

One last topic that we will cover concerns the zeros of random sums of orthogonal polynomials. These are polynomials of the form

$$P_n(x) = \sum_{k=0}^{n} Z_k p_k(x),$$

where $Z_0, Z_1, \ldots$ is a sequence of independent complex Gaussians, with mean zero and variance one, and $p_0, p_1, \ldots$ represents a set of orthonormal polynomials. For a given set of orthonormal polynomials orthogonalized over a suitable domain Ω, and with a weight function satisfying certain restrictions, Shiffman and Zelditch [29] showed that the limiting distribution of zeros is given by the equilibrium measure for the set Ω. We will formulate this result in more detail in Chapter 5.

1.6 Background Material

1.6.1 Spectral Density

We will now discuss some results that are variously attributed to Bochner, Herglotz, and Khinchine (see [5, 7]). Given a stationary sequence of Gaussian random variables, $X_0, X_1, \ldots$, the covariance function, $\Gamma(k)$, can be expressed as

$$\mathrm{E}\left[X_0 X_k\right] = \Gamma(k) = \int_{-\pi}^{\pi} e^{-ik\phi} F(d\phi),$$

where $F(\phi)$ is real, never-decreasing, and bounded. Furthermore, if $F(\phi)$ is also absolutely continuous, we have the formula

$$\Gamma(k) = \int_{-\pi}^{\pi} e^{-ik\phi} f(\phi) d\phi, \tag{1.2}$$

where $f(\phi)$ is called the spectral density of the covariance function.

A sufficient condition for the existence of $f(\phi)$ is that $\Gamma(k)$ is absolutely summable. Additionally, in this case it will be non-negative, continuous, and of the form

$$f(\phi) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \Gamma(k) e^{ik\phi}. \tag{1.3}$$

One example where the spectral density can be explicitly computed is in the case of an exponentially decaying covariance; that is, $\Gamma(k) = \rho^{|k|}$, where $\rho \in (0,1)$. Here, the spectral density will have the form

$$f(\phi) = \frac{1}{2\pi}\frac{1-\rho^2}{1-2\rho\cos\phi+\rho^2}. \tag{1.4}$$

This formulation of the covariance function using the spectral density will prove crucial for the majority of the results in this dissertation. Whenever some dependence is assumed among the coefficients of a random polynomial, it is highly likely that the ensuing computations will be more involved than in the independent case. For certain covariance functions, such as the constant and the exponentially decaying cases, there are ways of getting around this difficulty. For many other examples, however, this is not easily done. It is here that the spectral density will allow us to make the needed computations, and to derive the necessary asymptotic values. Furthermore, since we will only make some general assumptions on the spectral density, the results will also hold for a wide class of covariance functions. To shed a little more light on the behavior of the spectral density, we have included several graphs of the spectral density for various choices of $\Gamma(k)$ in Appendix A.

1.6.2 Kac-Rice Formula

One other result that will be important for our work is the modern formulation of the Kac-Rice formula. For $P_n(x)$, define

$$\begin{aligned} A(x) &:= \mathrm{E}\left[P_n^2(x)\right], & K(x) &:= \mathrm{E}\left[P_n(x)\right], \\ B(x) &:= \mathrm{E}\left[P_n(x)P_n'(x)\right], & K'(x) &:= \mathrm{E}\left[P_n'(x)\right], \\ C(x) &:= \mathrm{E}\left[(P_n'(x))^2\right]. \end{aligned}$$

We then have

Theorem 1.6.1 (Kac-Rice Formula)**.** *For $P_n(x)$, on a given interval (α,β) we have*

$$\begin{aligned} \mathrm{E}\left[N(\alpha,\beta)\right] = {} & \frac{1}{\pi}\int_\alpha^\beta \frac{\sqrt{AC-B^2}}{A}\exp\left(-\frac{K^2C+K'^2A-2K'KB}{2\left(AC-B^2\right)}\right)dx \\ & + \frac{1}{\pi}\int_\alpha^\beta \frac{\sqrt{2}|K'A^2-BK|}{A^{3/2}}\exp\left(-\frac{K^2}{2A}\right) erf\left(\frac{|K'A-BK|}{\sqrt{2A\left(AC-B^2\right)}}\right)dx. \end{aligned}$$

For a derivation see [13]. We will use the Kac-Rice formula for our work on the real zeros in Chapter 2, and the K-level crossings in Chapter 3.

Chapter 2

Real Zeros

From Section 1.6.1 we have seen that the behavior of the real zeros changes when a constant covariance is assumed among the coefficients. On the other hand, the behavior stays the same for an exponentially decaying covariance. A further question that could be asked is whether or not this result extends to a wider class of covariance functions, where the decay rates are between those considered by Sambandham. This chapter will be concerned with computing the expected number of real zeros of $P_n(x)$ for a general class of covariance functions.

2.1 Statement of Main Result

In what follows, we will show that, assuming the spectral density is both positive and continuous, the same asymptotic value for the expected number of real zeros holds as in the independent case. If we assume further that the spectral density has one continuous derivative, we will be able to derive an explicit value for the order of the error term. Noting that absolute summability of the covariance function guarantees the continuity of the spectral density, we will then see that behavior similar to the independent case can be expected for covariance functions with a wide range of decay rates. Let $C([-\pi, \pi])$ be the set of continuous functions on $[-\pi, \pi]$, while $C^1([-\pi, \pi])$ denotes the set of functions on $[-\pi, \pi]$ with one continuous derivative. The main result is then stated as follows.

Theorem 2.1.1. *Let $P_n(x)$ be the polynomial given in* (1)*, where the coefficients $X_0, X_1, \ldots$ form a stationary sequence of standard normals, with covariance function $\Gamma(k)$ and spectral density $f(\phi)$. Assume that the spectral density does not vanish. Letting $N(\alpha, \beta)$ be the number of zeros of $P_n(x)$ in the interval (α, β), it follows that*

$$\mathrm{E}[N(-\infty,\infty)] \sim \frac{2}{\pi}\log n, \qquad \textit{for } f \in C\left([-\pi,\pi]\right),$$
$$\mathrm{E}[N(-\infty,\infty)] = \frac{2}{\pi}\log n + O\left(\log\log n\right), \qquad \textit{for } f \in C^1\left([-\pi,\pi]\right),$$

as $n \to \infty$.

2.2 Deriving Upper Bounds

Before we proceed any further, a comment must be made. If we consider the function

$$\begin{aligned} x^n P_n\left(\frac{1}{x}\right) &= x^n\left(X_0 + X_1\frac{1}{x} + \cdots + X_n\frac{1}{x^n}\right) \\ &= X_0x^n + X_1x^{n-1} + \cdots + X_n, \end{aligned}$$

it can be seen that whenever there is a zero of $x^nP_n\left(\frac{1}{x}\right)$ in $(1,\infty)$, there is also a zero of $P_n(x)$ in $(0,1)$. Thus, since the distributions of the zeros of the two functions are the same, it is sufficient to only look at the interval from $(0,1)$. A similar argument works for the negative real line and allows us to restrict our analysis to the interval $(-1,1)$. By then taking twice the result, we will have a value for the total expected number of real zeros.

Our proof will loosely follow that of Sambandham, with some necessary modifications to account for the more general assumptions made on the coefficients. The first step is to show that there is a negligible amount of zeros on the intervals $(0, 1-\frac{1}{\log n})$, $(1-\frac{\log\log n}{n}, 1)$, $(-1+\frac{1}{\log n}, 0)$, and $(-1, -1+\frac{\log\log n}{n})$. Following this, we will then show that the number of real zeros in the intervals $(1-\frac{1}{\log n}, 1-\frac{\log\log n}{n})$ and $(-1+\frac{\log\log n}{n}, -1+\frac{1}{\log n})$ are each on the order of $\frac{1}{2\pi}\log n$. One significant difference from Sambandham's work is that rather than approximating the number of zeros with a specially chosen function, we will instead derive the asymptotic behavior directly from the Kac-Rice formula using the spectral density formulation of the covariance function. The result is that the number of zeros in $(-1,1)$ is on the order of $\frac{1}{\pi}\log n$ and, from the comments above, it follows that the total expected number of real zeros is on the order of $\frac{2}{\pi}\log n$.

Now, recalling that $N(\alpha,\beta)$ is the number of zeros of $P_n(x)$ in the interval (α,β), the Kac-Rice formula [13] gives the expected number of zeros on this interval as

$$\mathrm{E}[N(\alpha,\beta)] = \frac{1}{\pi}\int_{\alpha}^{\beta} \frac{\sqrt{AC-B}}{A} dx, \tag{2.1}$$

where

$$A(x) = \mathrm{E}\left[P_n^2(x)\right] = \sum_{k=0}^{n}\sum_{j=0}^{n}\Gamma(k-j)x^{k+j},$$

$$B(x) = \mathrm{E}\left[P_n(x)P_n'(x)\right] = \sum_{k=0}^{n}\sum_{j=0}^{n}\Gamma(k-j)kx^{k+j-1},$$

$$C(x) = \mathrm{E}\left[(P_n'(x))^2\right] = \sum_{k=0}^{n}\sum_{j=0}^{n}\Gamma(k-j)kjx^{k+j-2}.$$

Applying (1.2) we can rewrite these as

$$A = \int_{-\pi}^{\pi}\sum_{k=0}^{n}\sum_{j=0}^{n} e^{-i(k-j)\phi}x^{k+j}f(\phi)d\phi,$$

$$B = \int_{-\pi}^{\pi}\sum_{k=0}^{n}\sum_{j=0}^{n} e^{-i(k-j)\phi}kx^{k+j-1}f(\phi)d\phi,$$

$$C = \int_{-\pi}^{\pi}\sum_{k=0}^{n}\sum_{j=0}^{n} e^{-i(k-j)\phi}kjx^{k+j-2}f(\phi)d\phi.$$

We are now ready to prove our first lemma.

Lemma 2.2.1. *For the intervals* $(-1,-1+\frac{\log\log n}{n})$, $(-1+\frac{1}{\log n},0)$, $(0,1-\frac{1}{\log n})$, *and* $(1-\frac{\log\log n}{n},1)$, *the expected number of zeros is* $O(\log\log n)$.

Proof. Following the method of Sambandham, we will define the function

$$\begin{aligned}
H(x,y) &= \sum_{k=0}^{n}\sum_{j=0}^{n} e^{-i(k-j)\phi}x^k y^j \\
&= \sum_{k=0}^{n} e^{-ik\phi}x^k \sum_{j=0}^{n} e^{ij\phi}y^j \\
&= \frac{1-x^{n+1}e^{-i(n+1)\phi}}{1-xe^{-i\phi}} \cdot \frac{1-y^{n+1}e^{i(n+1)\phi}}{1-ye^{i\phi}},
\end{aligned}$$

to assist in our computations. Plugging into our formula for A leads to the expression

$$\begin{aligned}
A &= \int_{-\pi}^{\pi} H(x,x) f(\phi) d\phi \\
&= \int_{-\pi}^{\pi} \frac{1-x^{n+1}e^{-i(n+1)\phi}}{1-xe^{-i\phi}} \cdot \frac{1-x^{n+1}e^{i(n+1)\phi}}{1-xe^{i\phi}} f(\phi) d\phi.
\end{aligned} \tag{2.2}$$

Similarly,

$$\begin{aligned}
B &= \int_{-\pi}^{\pi} \left[\frac{\partial H(x,y)}{\partial y}\right]_{y=x} f(\phi) d\phi \\
&= \int_{-\pi}^{\pi} \left(\frac{1-x^{n+1}e^{-i(n+1)\phi}}{1-xe^{-i\phi}}\right) \\
&\quad \cdot \left(\frac{-(n+1)x^n e^{i(n+1)\phi}\left(1-xe^{i\phi}\right) - \left(1-x^{n+1}e^{i(n+1)\phi}\right)\left(-e^{i\phi}\right)}{\left(1-xe^{i\phi}\right)^2}\right) f(\phi) d\phi,
\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}
C &= \int_{-\pi}^{\pi} \left[\frac{\partial^2 H(x,y)}{\partial x \partial y}\right]_{y=x} f(\phi) d\phi \\
&= \int_{-\pi}^{\pi} \left(\frac{-(n+1)x^n e^{-i(n+1)\phi}\left(1-xe^{-i\phi}\right) - \left(1-x^{n+1}e^{-i(n+1)\phi}\right)\left(-e^{-i\phi}\right)}{\left(1-xe^{-i\phi}\right)^2}\right) \\
&\quad \cdot \left(\frac{-(n+1)x^n e^{i(n+1)\phi}\left(1-xe^{i\phi}\right) - \left(1-x^{n+1}e^{i(n+1)\phi}\right)\left(-e^{i\phi}\right)}{\left(1-xe^{i\phi}\right)^2}\right) f(\phi) d\phi.
\end{aligned} \tag{2.4}$$

For $x \in (0, 1 - \frac{1}{\log n})$ we have

$$A \sim \int_{-\pi}^{\pi} \frac{1}{(1-xe^{-i\phi})(1-xe^{i\phi})} f(\phi) d\phi,$$

and

$$C \sim \int_{-\pi}^{\pi} \frac{1}{(1-xe^{-i\phi})^2(1-xe^{i\phi})^2} f(\phi) d\phi$$

$$\leq \frac{1}{(1-x)^2}\int_{-\pi}^{\pi}\frac{1}{(1-xe^{-i\phi})(1-xe^{i\phi})}f(\phi)d\phi.$$

If we consider the quotient $\frac{\sqrt{AC-B^2}}{A}$, then

$$\frac{\sqrt{AC-B^2}}{A} < \left(\frac{C}{A}\right)^{1/2} \leq \frac{1}{1-x}.$$

Plugging into (2.1), it follows that

$$\begin{aligned} \mathrm{E}\left[N\left(0,1-\frac{1}{\log n}\right)\right] &= \int_0^{1-\frac{1}{\log n}}\frac{\sqrt{AC-B^2}}{A}dx \\ &\leq \int_0^{1-\frac{1}{\log n}}\frac{1}{1-x}dx \\ &= \log\log n. \end{aligned} \tag{2.5}$$

Thus,

$$\mathrm{E}\left[N\left(0,1-\frac{1}{\log n}\right)\right] = O\left(\log\log n\right).$$

To handle the interval $(-1+\frac{1}{\log n},0)$ we will substitute in $-x$, where $x\in(0,1-\frac{1}{\log n})$, to get

$$\begin{aligned} A &= \int_{-\pi}^{\pi} H(-x,-x)f(\phi)d\phi \\ &= \int_{-\pi}^{\pi}\frac{1-(-x)^{n+1}e^{-i(n+1)\phi}}{1+xe^{-i\phi}}\cdot\frac{1-(-x)^{n+1}e^{i(n+1)\phi}}{1+xe^{i\phi}}f(\phi)d\phi \\ &\sim \int_{-\pi}^{\pi}\frac{1}{(1+xe^{-i\phi})(1+xe^{i\phi})}f(\phi)d\phi. \end{aligned}$$

Likewise,

$$\begin{aligned} C &\sim \int_{-\pi}^{\pi}\frac{1}{\left(1+xe^{-i\phi}\right)^2\left(1+xe^{i\phi}\right)^2}f(\phi)d\phi \\ &\leq \frac{1}{(1-x)^2}\int_{-\pi}^{\pi}\frac{1}{\left(1+xe^{-i\phi}\right)\left(1+xe^{i\phi}\right)}f(\phi)d\phi. \end{aligned}$$

Considering the quotient $\frac{\sqrt{AC-B^2}}{A}$ once more gives us

$$\frac{\sqrt{AC-B^2}}{A} < \left(\frac{C}{A}\right)^{1/2} \leq \frac{1}{1-x},$$

just as before. Applying (2.1), we then have

$$\begin{aligned}\mathrm{E}\left[N\left(-1+\frac{1}{\log n},0\right)\right] &= \int_{-1+\frac{1}{\log n}}^{0} \frac{\sqrt{AC-B^2}}{A}dx \\ &\leq \int_{0}^{1-\frac{1}{\log n}} \frac{1}{1-x}dx \\ &= \log\log n.\end{aligned} \tag{2.6}$$

Thus,

$$\mathrm{E}\left[N\left(-1+\frac{1}{\log n},0\right)\right] = O\left(\log\log n\right).$$

Finally, we will consider the intervals $(-1,-1+\frac{\log\log n}{n})$ and $(1-\frac{\log\log n}{n},1)$. For $x \in (1-\frac{\log\log n}{n},1)$ we have the inequality

$$\begin{aligned}\frac{\sqrt{AC-B^2}}{A} &< \left(\frac{C}{A}\right)^{1/2} \\ &< \left(\frac{n^2\sum_{k=0}^{n}\sum_{j=0}^{n}\Gamma(k-j)x^{k+j-2}}{x^2\sum_{k=0}^{n}\sum_{j=0}^{n}\Gamma(k-j)x^{k+j-2}}\right)^{1/2} \\ &< cn.\end{aligned}$$

Thus,

$$\begin{aligned}\mathrm{E}\left[N\left(1-\frac{\log\log n}{n},1\right)\right] &= \int_{1-\frac{\log\log n}{n}}^{1} \frac{\sqrt{AC-B^2}}{A}dx \\ &\leq \int_{1-\frac{\log\log n}{n}}^{1} cndx \\ &= O(\log\log n).\end{aligned} \tag{2.7}$$

For $(-1,-1+\frac{\log\log n}{n})$ we will substitute in $-x$. Notice that since $f(\phi)$ is continuous and non-zero on $[-\pi,\pi]$, we can bound it from below by a constant $\frac{m}{2\pi}>0$. We then have

$$\begin{aligned}A &\geq \frac{m}{2\pi}\int_{-\pi}^{\pi} \frac{1-(-x)^{n+1}e^{-i(n+1)\phi}}{1+xe^{-i\phi}} \cdot \frac{1-(-x)^{n+1}e^{i(n+1)\phi}}{1+xe^{i\phi}}d\phi \\ &= m\sum_{k=0}^{n}(-x)^{2k},\end{aligned}$$

since $f(\phi) \equiv \frac{1}{2\pi}$ in the independent case. Similarly, we can bound $f(\phi)$ from above by $\frac{M}{2\pi} < \infty$. This yields the inequality

$$\begin{aligned}
C \leq \frac{M}{2\pi} \int_{-\pi}^{\pi} & \frac{-(n+1)(-x)^n e^{-i(n+1)\phi}\left(1+xe^{-i\phi}\right) + \left(1-(-x)^{n+1}e^{-i(n+1)\phi}\right)e^{-i\phi}}{(1+xe^{-i\phi})^2} \\
& \cdot \frac{-(n+1)(-x)^n e^{i(n+1)\phi}\left(1+xe^{i\phi}\right) + \left(1-(-x)^{n+1}e^{i(n+1)\phi}\right)e^{i\phi}}{(1+xe^{i\phi})^2} d\phi. \\
= M \sum_{k=0}^{n} & k^2(-x)^{2k-2} \\
\leq M & \frac{n^2}{(-x)^2} \sum_{k=0}^{n} (-x)^{2k}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{\sqrt{AC-B^2}}{A} &< \left(\frac{C}{A}\right)^{1/2} \\
&\leq \left(\frac{Mn^2}{mx^2}\right)^{1/2} \\
&< cn.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathrm{E}\left[N\left(-1,-1+\frac{\log\log n}{n}\right)\right] &= \int_{-1}^{-1+\frac{\log\log n}{n}} \frac{\sqrt{AC-B^2}}{A}dx \\
&\leq \int_{1-\frac{\log\log n}{n}}^{1} cndx \qquad (2.8) \\
&= O(\log\log n).
\end{aligned}$$

Combining (2.5)-(2.8), the result then follows. □

2.3 Computing Zeros on $(-1, 1)$

Now that we have found an upper bound for the expected number of zeros in our initial four intervals, we will spend the rest of our time deriving explicit values for the intervals $(-1+\frac{\log\log n}{n}, -1+\frac{1}{\log n})$ and $(1-\frac{1}{\log n}, 1-\frac{\log\log n}{n})$. As stated before, this will be done by deriving asymptotic values for the expressions given in (2.1). We will formulate this result as an additional lemma.

Lemma 2.3.1. *The expected number of zeros for the polynomial $P_n(x)$ in each of the intervals $(-1+\frac{\log\log n}{n}, -1+\frac{1}{\log n})$ and $(1-\frac{1}{\log n}, 1-\frac{\log\log n}{n})$ is given by the following:*

(i) For $f \in C\left([-\pi,\pi]\right)$,

$$\mathrm{E}\left[N\left(-1+\frac{\log\log n}{n}, -1+\frac{1}{\log n}\right)\right] = \mathrm{E}\left[N\left(1-\frac{1}{\log n}, 1-\frac{\log\log n}{n}\right)\right] \sim \frac{1}{2\pi}\log n.$$

(ii) For $f \in C^1\left([-\pi,\pi]\right)$,

$$\mathrm{E}\left[N\left(-1+\frac{\log\log n}{n}, -1+\frac{1}{\log n}\right)\right] = \mathrm{E}\left[N\left(1-\frac{1}{\log n}, 1-\frac{\log\log n}{n}\right)\right] = \frac{1}{2\pi}\log n + O\left(\log\log n\right).$$

Proof. Consider the interval $(1-\frac{1}{\log n}, 1-\frac{\log\log n}{n})$. For $x = 1-y$, define $g(y) = y\frac{\log n}{\log\log n}$. Also, let $M > 0$ be chosen such that $f(\phi) \le M$, for any $\phi \in [-\pi,\pi]$. Recalling (2.1), for A we have

$$\begin{aligned} A &= \int_{-\pi}^{\pi} \frac{1}{(1-xe^{-i\phi})(1-xe^{i\phi})} f(\phi)d\phi \\ &\quad + x^{n+1}\int_{-\pi}^{\pi} \frac{-\left(e^{-i(n+1)\phi}+e^{i(n+1)\phi}\right)+x^{n+1}}{(1-xe^{-i\phi})(1-xe^{i\phi})} f(\phi)d\phi \\ &= A_1 + O\left(x^{n+1}A_1\right). \end{aligned} \tag{2.9}$$

Our next step is to determine the asymptotic value for A_1. We have,

$$A_1 = 2\int_0^{g(y)} \frac{1}{1-2x\cos\phi+x^2} f(\phi)d\phi + 2\int_{g(y)}^{\pi} \frac{1}{1-2x\cos\phi+x^2} f(\phi)d\phi.$$

Looking at the first integral yields

$$\begin{aligned} &= 2\int_0^{g(y)} \frac{f(\phi)}{1-2(1-y)\left(1-\frac{\phi^2}{2}+O\left(\phi^4\right)\right)+(1-y)^2} d\phi \\ &= 2\int_0^{g(y)} \frac{f(\phi)}{\phi^2+y^2-y\phi^2+O\left(\phi^4\right)} d\phi \\ &= 2\int_0^{g(y)} \frac{f(\phi)}{\phi^2+y^2} d\phi + O(1). \end{aligned}$$

For $f \in C([-\pi, \pi])$ this becomes

$$\begin{aligned}
&\sim 2\int_0^{g(y)} \frac{f(0)}{\phi^2 + y^2} d\phi \\
&= \frac{2f(0)}{y} \arctan\left(\frac{g(y)}{y}\right),
\end{aligned}$$

while for $f \in C^1([-\pi, \pi])$ we have the more detailed representation

$$\begin{aligned}
&= 2\int_0^{g(y)} \frac{f(0) + f'(\phi_0)\phi}{\phi^2 + y^2} d\phi + O(1), \quad (\text{where } \phi_0 \in (0, \phi)) \\
&= \frac{2f(0)}{y} \arctan\left(\frac{g(y)}{y}\right) + O\left(\int_0^{g(y)} \frac{\phi}{\phi^2 + y^2} d\phi\right) \\
&= \frac{2f(0)}{y} \arctan\left(\frac{g(y)}{y}\right) + O(\log\log n).
\end{aligned}$$

For the second integral in A_1,

$$\begin{aligned}
&= \int_{g(y)}^{(g(y))^{1/3}} \frac{2}{1 - 2x\cos\phi + x^2} f(\phi) d\phi + \int_{(g(y))^{1/3}}^{\pi} \frac{2}{1 - 2x\cos\phi + x^2} f(\phi) d\phi \\
&\sim \int_{g(y)}^{(g(y))^{1/3}} \frac{2f(0)}{y^2 + \phi^2} d\phi + \int_{(g(y))^{1/3}}^{\pi} \frac{2}{1 - 2x\cos\phi + x^2} f(\phi) d\phi \\
&\leq \int_{g(y)}^{(g(y))^{1/3}} \frac{2f(0)}{\phi^2} d\phi + \int_{(g(y))^{1/3}}^{\pi} \frac{2M}{1 - 2x\cos\left((g(y))^{1/3}\right) + x^2} d\phi \\
&\sim 2f(0)\left(\frac{1}{g(y)} - \frac{1}{(g(y))^{1/3}}\right) + \frac{2M\pi}{(g(y))^{2/3} + y^2} \\
&= O\left(\frac{1}{g(y)}\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
A_1 &\sim \frac{2f(0)}{y} \arctan\left(\frac{g(y)}{y}\right), \qquad \text{for } f \in C([-\pi, \pi]), \\
A_1 &= \frac{2f(0)}{y} \arctan\left(\frac{g(y)}{y}\right) + O\left(\frac{1}{g(y)}\right), \qquad \text{for } f \in C^1([-\pi, \pi]).
\end{aligned}$$

Now, notice that

$$\begin{aligned}
x^{n+1} A_1 &\sim (1 - y)^{n+1} \frac{c}{y} \\
&\leq \left(1 - \frac{\log\log n}{n}\right)^{n+1} \frac{c}{y}
\end{aligned}$$

$$\sim \frac{c}{y \log n}$$
$$= o\left(\frac{1}{g(y)}\right).$$

Combining these results and plugging into (2.9) yields

$$\begin{aligned}
A &\sim \frac{2f(0)}{y} \arctan\left(\frac{g(y)}{y}\right), \qquad \text{for } f \in C\left([-\pi,\pi]\right), \\
A &= \frac{2f(0)}{y} \arctan\left(\frac{g(y)}{y}\right) + O\left(\frac{1}{g(y)}\right), \qquad \text{for } f \in C^1\left([-\pi,\pi]\right).
\end{aligned} \tag{2.10}$$

Considering B next,

$$\begin{aligned}
B &= \int_{-\pi}^{\pi} \frac{e^{i\phi}}{(1-xe^{-i\phi})(1-xe^{i\phi})^2} f(\phi)d\phi \\
&\quad + \int_{-\pi}^{\pi} \frac{-(n+1)x^n e^{i(n+1)\phi}\left(1-xe^{i\phi}\right)}{(1-xe^{-i\phi})(1-xe^{i\phi})^2} f(\phi)d\phi \\
&\quad + x^{n+1} \int_{-\pi}^{\pi} \frac{(n+1)x^n\left(1-xe^{i\phi}\right) - e^{-in\phi}\left(1-x^{n+1}e^{i(n+1)\phi}\right) - e^{i(n+2)\phi}}{(1-xe^{-i\phi})(1-xe^{i\phi})^2} f(\phi)d\phi \\
&= B_1 + B_2 + O\left(x^{n+1}\left(|B_1| + |B_2|\right)\right).
\end{aligned}$$

Since we will end up showing that the B_1 term dominates, we can rewrite this as

$$B = B_1 + B_2 + O\left(x^{n+1}B_1\right). \tag{2.11}$$

To analyze B_1 we will split it into two integrals,

$$\begin{aligned}
B_1 &= 2\int_0^{g(y)} \frac{\cos\phi - 1 + y}{\left(1-2(1-y)\cos\phi + (1-y)^2\right)^2} f(\phi)d\phi \\
&\quad + 2\int_{g(y)}^{\pi} \frac{\cos\phi - 1 + y}{\left(1-2(1-y)\cos\phi + (1-y)^2\right)^2} f(\phi)d\phi.
\end{aligned}$$

For the first integral we have

$$\begin{aligned}
&= 2\int_0^{g(y)} \frac{y - \frac{\phi^2}{2} + O\left(\phi^4\right)}{\left(1-2(1-y)\left(1-\frac{\phi^2}{2}+O\left(\phi^4\right)\right)+(1-y)^2\right)^2} f(\phi)d\phi \\
&= 2\int_0^{g(y)} \frac{y}{\left(\phi^2+y^2-y\phi^2+O\left(\phi^4\right)\right)^2} f(\phi)d\phi + O\left(\frac{1}{y}\right) \\
&= 2\int_0^{g(y)} \frac{y}{\left(\phi^2+y^2\right)^2} f(\phi)d\phi + O\left(\frac{1}{y}\right).
\end{aligned}$$

For $f \in C([-\pi,\pi])$ this becomes

$$\begin{aligned}
&\sim 2\int_0^{g(y)} \frac{yf(0)}{\left(\phi^2+y^2\right)^2} d\phi \\
&= \frac{f(0)}{y^2}\left[\frac{g(y)y}{g^2(y)+y^2} + \arctan\left(\frac{g(y)}{y}\right)\right] \\
&\sim \frac{f(0)}{y^2}\arctan\left(\frac{g(y)}{y}\right),
\end{aligned}$$

while for $f \in C^1([-\pi,\pi])$

$$\begin{aligned}
&= 2\int_0^{g(y)} \frac{yf(0)+yf'(\phi_0)\phi}{\left(\phi^2+y^2\right)^2} d\phi + O\left(\frac{1}{y}\right) \quad \text{(where } \phi_0 \in (0,\phi)\text{)} \\
&= \frac{f(0)}{y^2}\left[\frac{\phi y}{\phi^2+y^2} + \arctan\left(\frac{\phi}{y}\right)\right]\Bigg|_0^{g(y)} + O\left(\int_0^{g(y)} \frac{y\phi}{\left(\phi^2+y^2\right)^2} d\phi + \frac{1}{y}\right) \\
&= \frac{f(0)}{y^2}\left[\frac{g(y)y}{g^2(y)+y^2} + \arctan\left(\frac{g(y)}{y}\right)\right] + O\left(\frac{1}{y}\right) \\
&= \frac{f(0)}{y^2}\arctan\left(\frac{g(y)}{y}\right) + O\left(\frac{1}{yg(y)}\right).
\end{aligned}$$

For the second integral in B_1,

$$\begin{aligned}
&\left|2\int_{g(y)}^{\pi} \frac{\cos\phi - x}{(1-2x\cos\phi+x^2)^2} f(\phi) d\phi\right| \\
&\le 2\int_{g(y)}^{(g(y))^{1/3}} \frac{|\cos\phi - x|\, f(\phi)}{(1-2x\cos\phi+x^2)^2} d\phi + 2\int_{(g(y))^{1/3}}^{\pi} \frac{|\cos\phi - x|\, f(\phi)}{(1-2x\cos\phi+x^2)^2} d\phi \\
&\sim 2f(0)\int_{g(y)}^{(g(y))^{1/3}} \frac{\left|y - \frac{\phi^2}{2} + O\left(\phi^4\right)\right|}{\left(\phi^2+y^2\right)^2} d\phi + 2\int_{(g(y))^{1/3}}^{\pi} \frac{|\cos\phi - x|\, f(\phi)}{(1-2x\cos\phi+x^2)^2} d\phi \\
&\le 2f(0)\int_{g(y)}^{(g(y))^{1/3}} \frac{y+\phi^2+O\left(\phi^4\right)}{\phi^4} d\phi + \int_{(g(y))^{1/3}}^{\pi} \frac{4M}{(1-2x\cos(g(y))^{1/3}+x^2)^2} d\phi \\
&\sim \frac{2f(0)y}{3}\left(\frac{1}{(g(y))^3} - \frac{1}{g(y)}\right) + 2f(0)\left(\frac{1}{g(y)} - \frac{1}{(g(y))^{1/3}}\right) + \frac{4\pi M}{(g(y))^{4/3}} \\
&= O\left(\frac{y}{(g(y))^3}\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
B_1 &\sim \frac{f(0)}{y^2}\arctan\left(\frac{g(y)}{y}\right), \qquad \text{for } f \in C([-\pi,\pi]), \\
B_1 &= \frac{f(0)}{y^2}\arctan\left(\frac{g(y)}{y}\right) + O\left(\frac{1}{yg(y)}\right), \qquad \text{for } f \in C^1([-\pi,\pi]).
\end{aligned}$$

For B_2 we have,

$$\begin{aligned}
B_2 &= \left| \int_{-\pi}^{\pi} \frac{-(n+1)x^n e^{i(n+1)\phi}(1-xe^{i\phi})}{(1-xe^{-i\phi})(1-xe^{i\phi})^2} f(\phi) d\phi \right| \\
&\leq \int_{-\pi}^{\pi} \frac{(n+1)x^n}{(1-xe^{-i\phi})(1-xe^{i\phi})} f(\phi) d\phi \\
&\sim c(n+1)\frac{(1-y)^n}{y},
\end{aligned}$$

where the last line follows from our work on A. Now, notice that for $y = \frac{\log\log n}{n}$ we have

$$(n+1)\frac{(1-y)^n}{y} \sim \frac{n^2}{\log n \log\log n} = \frac{1}{yg(y)}.$$

Since $(1-y)^n y$ is a decreasing function as y increases, it follows that for $y \in (\frac{\log\log n}{n}, \frac{1}{\log n})$,

$$(n+1)(1-y)^n y \leq \frac{y}{g(y)} \Rightarrow (n+1)\frac{(1-y)^n}{y} = O\left(\frac{1}{yg(y)}\right). \tag{2.12}$$

This also implies that

$$\begin{aligned}
x^{n+1}(B_1+B_2) &\sim c\frac{(1-y)^{n+1}}{y^2} \\
&< c(n+1)\frac{(1-y)^{n+1}}{y}.
\end{aligned}$$

Thus,

$$\begin{aligned}
B &\sim \frac{f(0)}{y^2}\arctan\left(\frac{g(y)}{y}\right), \qquad \text{for } f \in C\left([-\pi,\pi]\right), \\
B &= \frac{f(0)}{y^2}\arctan\left(\frac{g(y)}{y}\right) + O\left(\frac{1}{yg(y)}\right), \qquad \text{for } f \in C^1\left([-\pi,\pi]\right).
\end{aligned} \tag{2.13}$$

Turning now to C,

$$\begin{aligned}
C = &\int_{-\pi}^{\pi} \frac{1}{(1-xe^{-i\phi})^2(1-xe^{i\phi})^2} f(\phi) d\phi + \int_{-\pi}^{\pi} \frac{(n+1)^2 x^{2n}}{(1-xe^{-i\phi})(1-xe^{i\phi})} f(\phi) d\phi \\
&+ 2\int_{-\pi}^{\pi} \frac{-(n+1)x^n e^{-in\phi}\left(1-x^{n+1}e^{i(n+1)\phi}\right)}{(1-xe^{-i\phi})(1-xe^{i\phi})^2} f(\phi) d\phi \\
&+ x^{n+1}\int_{-\pi}^{\pi} \frac{-\left(e^{-i(n+1)\phi}+e^{i(n+1)\phi}\right)+x^{n+1}}{(1-xe^{-i\phi})^2(1-xe^{i\phi})^2} f(\phi) d\phi
\end{aligned}$$

$$= C_1 + C_2 + C_3 + O\left(x^{n+1} C_1\right).$$

Splitting up C_1 gives us

(2.14)

$$C_1 = 2\int_0^{g(y)} \frac{1}{(1-2x\cos\phi + x^2)^2} f(\phi) d\phi + 2\int_{g(y)}^{\pi} \frac{1}{(1-2x\cos\phi + x^2)^2} f(\phi) d\phi.$$

For the first term we have

$$\begin{aligned}
&= 2\int_0^{g(y)} \frac{1}{\left(1-2(1-y)\left(1-\frac{\phi^2}{2}+O\left(\phi^4\right)\right)+(1-y)^2\right)^2} f(\phi) d\phi \\
&= 2\int_0^{g(y)} \frac{1}{\left(\phi^2+y^2-y\phi^2+O\left(\phi^4\right)\right)^2} f(\phi) d\phi \\
&= 2\int_0^{g(y)} \frac{f(\phi)}{\left(\phi^2+y^2\right)^2} d\phi + O\left(\frac{1}{y^2}\right).
\end{aligned}$$

For $f \in C([-\pi,\pi])$ it follows that

$$\begin{aligned}
&\sim 2\int_0^{g(y)} \frac{f(0)}{\left(\phi^2+y^2\right)^2} d\phi \\
&= \frac{f(0)}{y^3}\left[\frac{g(y)y}{g^2(y)+y^2} + \arctan\left(\frac{g(y)}{y}\right)\right] \\
&\sim \frac{f(0)}{y^3}\arctan\left(\frac{g(y)}{y}\right),
\end{aligned}$$

while for $f \in C^1([-\pi,\pi])$

$$\begin{aligned}
&= 2\int_0^{g(y)} \frac{f(0)+f'(\phi_0)\phi}{\left(\phi^2+y^2\right)^2} d\phi + O\left(\frac{1}{y^2}\right), \quad (\text{where } \phi_0 \in (0,\phi)) \\
&= \frac{f(0)}{y^3}\left[\frac{\phi y}{\phi^2+y^2} + \arctan\left(\frac{\phi}{y}\right)\right]\Bigg|_0^{g(y)} + O\left(\int_0^{g(y)} \frac{\phi}{\left(\phi^2+y^2\right)^2} d\phi + \frac{1}{y^2}\right) \\
&= \frac{f(0)}{y^3}\left[\frac{g(y)y}{g^2(y)+y^2} + \arctan\left(\frac{g(y)}{y}\right)\right] + O\left(\frac{1}{y^2}\right) \\
&= \frac{f(0)}{y^3}\arctan\left(\frac{g(y)}{y}\right) + O\left(\frac{1}{y^2 g(y)}\right).
\end{aligned}$$

For the second integral in C_1 we have

$$
\begin{aligned}
&= 2\int_{g(y)}^{(g(y))^{1/3}} \frac{f(\phi)}{\left(1-2x\cos\phi+x^2\right)^2}d\phi + 2\int_{(g(y))^{1/3}}^{\pi} \frac{f(\phi)}{\left(1-2x\cos\phi+x^2\right)^2}d\phi \\
&\sim 2f(0)\int_{g(y)}^{(g(y))^{1/3}} \frac{1}{\left(\phi^2+y^2\right)^2}d\phi + 2\int_{(g(y))^{1/3}}^{\pi} \frac{f(\phi)}{\left(1-2x\cos\phi+x^2\right)^2}d\phi \\
&\leq 2f(0)\int_{g(y)}^{(g(y))^{1/3}} \frac{1}{\phi^4}d\phi + 2M\int_{(g(y))^{1/3}}^{\pi} \frac{1}{\left(1-2x\cos\left((g(y))^{1/3}\right)+x^2\right)^2}d\phi \\
&\sim 2f(0)\left(\frac{1}{(g(y))^3}-\frac{1}{g(y)}\right)+\frac{2M\pi}{(g(y))^{4/3}} \\
&= O\left(\frac{1}{(g(y))^3}\right).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
C_1 &\sim \frac{f(0)}{y^3}\arctan\left(\frac{g(y)}{y}\right), \qquad \text{for } f\in C\left([-\pi,\pi]\right), \\
C_1 &= \frac{f(0)}{y^3}\arctan\left(\frac{g(y)}{y}\right)+O\left(\frac{1}{y^2 g(y)}\right), \qquad \text{for } f\in C^1\left([-\pi,\pi]\right).
\end{aligned} \tag{2.15}
$$

For C_2,

$$
\begin{aligned}
C_2 &= \int_{-\pi}^{\pi}\frac{(n+1)^2x^{2n}(1-xe^{i\phi})(1-xe^{-i\phi})}{(1-xe^{-i\phi})^2(1-xe^{i\phi})^2}f(\phi)d\phi \\
&\sim c(n+1)^2\frac{(1-y)^{2n}}{y},
\end{aligned}
$$

while for C_3 we have

$$
\begin{aligned}
|C_3| &\leq c\int_{-\pi}^{\pi}\left|\frac{-(n+1)x^n e^{-in\phi}}{(1-xe^{-i\phi})(1-xe^{i\phi})^2}f(\phi)\right|d\phi \\
&\leq c(C_1C_2)^{1/2} \quad \text{(by Cauchy-Schwarz)} \\
&\sim c\frac{(n+1)(1-y)^n}{y^2}.
\end{aligned}
$$

Also, note that

$$
x^{n+1}C_1 \sim c\frac{(1-y)^{n+1}}{y^3} \leq \frac{(n+1)(1-y)^n}{y^2}.
$$

From (2.12) we know that $(n+1)(1-y)^n = O\left(1/g(y)\right)$, which implies

$$
\begin{aligned}
\frac{(n+1)(1-y)^n}{y^2} &= O\left(\frac{1}{y^2 g(y)}\right), \\
\frac{(n+1)^2(1-y)^{2n}}{y} &= O\left(\frac{1}{y g^2(y)}\right).
\end{aligned} \tag{2.16}
$$

Thus, we can conclude that

$$
\begin{aligned}
C &\sim \frac{f(0)}{y^3}\arctan\left(\frac{g(y)}{y}\right), \qquad \text{for } f \in C\left([-\pi,\pi]\right), \\
C &= \frac{f(0)}{y^3}\arctan\left(\frac{g(y)}{y}\right) + O\left(\frac{1}{y^2 g(y)}\right), \qquad \text{for } f \in C^1\left([-\pi,\pi]\right).
\end{aligned} \tag{2.17}
$$

Combining (2.10), (2.13), and (2.17), for $f \in C\left([-\pi,\pi]\right)$ we then have

$$
\begin{aligned}
\frac{\sqrt{AC-B^2}}{A} &\sim \sqrt{\frac{\pi f(0)}{y}\cdot\frac{\pi f(0)}{2y^3} - \left(\frac{\pi f(0)}{2y^2}\right)^2}\left(\frac{\pi f(0)}{y}\right)^{-1} \\
&= \frac{1}{2y}.
\end{aligned}
$$

For $f \in C^1\left([-\pi,\pi]\right)$ this becomes

$$
\begin{aligned}
AC - B^2 &= \\
&\left[\frac{2f(0)}{y}\arctan\left(\frac{g(y)}{y}\right) + O\left(\frac{1}{g(y)}\right)\right]\left[\frac{f(0)}{y^3}\arctan\left(\frac{g(y)}{y}\right) + O\left(\frac{1}{y^2 g(y)}\right)\right] \\
&\quad - \left[\frac{f(0)}{y^2}\arctan\left(\frac{g(n)}{y}\right) + O\left(\frac{1}{y g(y)}\right)\right]^2 \\
&= \frac{f^2(0)}{y^4}\left[\arctan\left(\frac{g(y)}{y}\right)\right]^2 + O\left(\frac{1}{y^3 g(y)}\right),
\end{aligned} \tag{2.18}
$$

and

$$
\begin{aligned}
\frac{\sqrt{AC-B^2}}{A} &= \frac{\sqrt{\frac{f^2(0)}{y^4}\left[\arctan\left(\frac{g(y)}{y}\right)\right]^2 + O\left(\frac{1}{y^3 g(y)}\right)}}{\frac{2f(0)}{y}\arctan\left(\frac{g(y)}{y}\right) + O\left(\frac{1}{g(y)}\right)} \\
&= \frac{\sqrt{\frac{f^2(0)}{y^4}\left[\arctan\left(\frac{g(y)}{y}\right)\right]^2 + O\left(\frac{1}{y^3 g(y)}\right)}}{\frac{2f(0)}{y}\arctan\left(\frac{g(y)}{y}\right)} + O\left(\frac{1}{g(y)}\right) \\
&= \frac{1}{2y} + O\left(\frac{1}{g(y)}\right).
\end{aligned} \tag{2.19}
$$

Plugging these into (2.1) gives us

$$
\mathrm{E}\left[N\left(1-\frac{1}{\log n}, 1-\frac{\log\log n}{n}\right)\right] = \frac{1}{\pi}\int_{1-\frac{1}{\log n}}^{1-\frac{\log\log n}{n}} \frac{\sqrt{AC-B^2}}{A}dx \tag{2.20}
$$

$$\sim \frac{1}{\pi}\int_{\frac{\log\log n}{n}}^{\frac{1}{\log n}} \frac{1}{2y}dy$$
$$= \frac{1}{2\pi}\log y\Big|_{\frac{\log\log n}{n}}^{\frac{1}{\log n}}$$
$$\sim \frac{1}{2\pi}\log n,$$

for $f \in C([-\pi,\pi])$, and

$$\begin{aligned}
\mathrm{E}\left[N\left(1-\frac{1}{\log n}, 1-\frac{\log\log n}{n}\right)\right] &= \frac{1}{\pi}\int_{1-\frac{1}{\log n}}^{1-\frac{\log\log n}{n}} \frac{\sqrt{AC-B^2}}{A}dx \\
&= \frac{1}{\pi}\int_{\frac{\log\log n}{n}}^{\frac{1}{\log n}} \left[\frac{1}{2y} + O\left(\frac{1}{g(y)}\right)\right] dy \\
&= \frac{1}{2\pi}\log y\Big|_{\frac{\log\log n}{n}}^{\frac{1}{\log n}} + O(\log\log n) \\
&= \frac{1}{2\pi}\log n + O(\log\log n),
\end{aligned} \tag{2.21}$$

for $f \in C^1([-\pi,\pi])$.

To handle the interval from $(-1+\frac{\log\log n}{n}, -1+\frac{1}{\log n})$ we will substitute in $-x = -1+y$, where $x \in (1-\frac{1}{\log n}, 1-\frac{\log\log n}{n})$. We then have

$$A = \int_{-\pi}^{\pi} \frac{1-(-x)^{n+1}e^{-i(n+1)\phi}}{1+xe^{-i\phi}} \cdot \frac{1-(-x)^{n+1}e^{i(n+1)\phi}}{1+xe^{i\phi}} f(\phi)d\phi,$$

$$B = \int_{-\pi}^{\pi} \left(\frac{1-(-x)^{n+1}e^{-i(n+1)\phi}}{1+xe^{-i\phi}}\right) \cdot \left(\frac{-(n+1)(-x)^n e^{i(n+1)\phi}\left(1+xe^{i\phi}\right) + \left(1-(-x)^{n+1}e^{i(n+1)\phi}\right)e^{i\phi}}{\left(1+xe^{i\phi}\right)^2}\right) f(\phi)d\phi,$$

and

$$C = \int_{-\pi}^{\pi} \left(\frac{-(n+1)(-x)^n e^{-i(n+1)\phi}\left(1+xe^{-i\phi}\right) + \left(1-(-x)^{n+1}e^{-i(n+1)\phi}\right)e^{-i\phi}}{\left(1+xe^{-i\phi}\right)^2}\right) \cdot \left(\frac{-(n+1)(-x)^n e^{i(n+1)\phi}\left(1+xe^{i\phi}\right) + \left(1-(-x)^{n+1}e^{i(n+1)\phi}\right)e^{i\phi}}{\left(1+xe^{i\phi}\right)^2}\right) f(\phi)d\phi.$$

Considering A first,

$$\begin{aligned}
A = &\int_{-\pi}^{\pi} \frac{1}{(1+xe^{-i\phi})(1+xe^{i\phi})} f(\phi)d\phi \\
&+ (-x)^{n+1} \int_{-\pi}^{\pi} \frac{-\left(e^{-i(n+1)\phi} + e^{i(n+1)\phi}\right) + (-x)^{n+1}}{(1+xe^{-i\phi})(1+xe^{i\phi})} f(\phi)d\phi \qquad (2.22)\\
= &A_1 + O\left((-x)^{n+1} A_1\right).
\end{aligned}$$

Splitting up A_1, we have

$$A_1 = 2\int_{\pi-g(y)}^{\pi} \frac{1}{1+2x\cos\phi + x^2} f(\phi)d\phi + 2\int_{0}^{\pi-g(y)} \frac{1}{1+2x\cos\phi + x^2} f(\phi)d\phi$$

The first integral becomes

$$\begin{aligned}
&= 2\int_{\pi-g(y)}^{\pi} \frac{f(\phi)}{1+2(1-y)\left(-1+\frac{(\phi-\pi)^2}{2} + O\left((\phi-\pi)^4\right)\right) + (1-y)^2} d\phi \\
&= 2\int_{\pi-g(y)}^{\pi} \frac{f(\phi)}{(\phi-\pi)^2 + y^2 - y(\phi-\pi)^2 + O\left((\phi-\pi)^4\right)} d\phi \\
&= 2\int_{\pi-g(y)}^{\pi} \frac{f(\phi)}{(\phi-\pi)^2 + y^2} d\phi + O(1).
\end{aligned}$$

For $f \in C([-\pi,\pi])$ this yields

$$\begin{aligned}
&\sim 2\int_{\pi-g(y)}^{\pi} \frac{f(\pi)}{(\phi-\pi)^2 + y^2} d\phi \\
&= \frac{2f(\pi)}{y} \arctan\left(\frac{g(y)}{y}\right),
\end{aligned}$$

while for $f \in C^1([-\pi,\pi])$ we have

$$\begin{aligned}
&= 2\int_{\pi-g(y)}^{\pi} \frac{f(\pi) + f'(\phi_0)(\phi-\pi)}{(\phi-\pi)^2 + y^2} d\phi + O(1), \quad (\text{where } \phi_0 \in (\pi-\phi,\pi)) \\
&= \frac{2f(\pi)}{y} \arctan\left(\frac{g(y)}{y}\right) + O\left(\int_{\pi-g(y)}^{\pi} \frac{\phi-\pi}{(\phi-\pi)^2 + y^2} d\phi\right) \\
&= \frac{2f(\pi)}{y} \arctan\left(\frac{g(y)}{y}\right) + O\left(\log\log n\right).
\end{aligned}$$

For the second integral in A_1,

$$= \int_{\pi-(g(y))^{1/3}}^{\pi-g(y)} \frac{2}{1+2x\cos\phi + x^2} f(\phi)d\phi + \int_{0}^{\pi-(g(y))^{1/3}} \frac{2}{1+2x\cos\phi + x^2} f(\phi)d\phi$$

$$
\begin{aligned}
&\sim \int_{\pi-(g(y))^{1/3}}^{\pi-g(y)} \frac{2f(\pi)}{y^2+(\phi-\pi)^2} d\phi + \int_0^{\pi-(g(y))^{1/3}} \frac{2}{1+2x\cos\phi+x^2} f(\phi) d\phi \\
&\le \int_{\pi-(g(y))^{1/3}}^{\pi-g(y)} \frac{2f(\pi)}{(\phi-\pi)^2} d\phi + \int_0^{\pi-(g(y))^{1/3}} \frac{2M}{1+2x\cos\left(\pi-(g(y))^{1/3}\right)+x^2} d\phi \\
&\sim 2f(\pi)\left(\frac{1}{g(y)} - \frac{1}{(g(y))^{1/3}}\right) + \frac{2M\pi}{(g(y))^{2/3}+y^2} \\
&= O\left(\frac{1}{g(y)}\right).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
A_1 &\sim \frac{2f(\pi)}{y} \arctan\left(\frac{g(y)}{y}\right), \qquad \text{for } f \in C\left([-\pi,\pi]\right), \\
A_1 &= \frac{2f(\pi)}{y} \arctan\left(\frac{g(y)}{y}\right) + O\left(\frac{1}{g(y)}\right), \qquad \text{for } f \in C^1\left([-\pi,\pi]\right).
\end{aligned}
$$

Next, notice that

$$
\begin{aligned}
|(-x)^{n+1}A_1| &\sim |(-1+y)^{n+1}| \frac{c}{y} \\
&\le \left(1 - \frac{\log\log n}{n}\right)^{n+1} \frac{c}{y} \\
&\sim \frac{c}{y\log n} \\
&= o\left(\frac{1}{g(y)}\right).
\end{aligned}
$$

Combining these results and plugging into (2.22) yields

$$
\begin{aligned}
A &\sim \frac{2f(\pi)}{y} \arctan\left(\frac{g(y)}{y}\right), \qquad \text{for } f \in C\left([-\pi,\pi]\right), \\
A &= \frac{2f(\pi)}{y} \arctan\left(\frac{g(y)}{y}\right) + O\left(\frac{1}{g(y)}\right), \qquad \text{for } f \in C^1\left([-\pi,\pi]\right).
\end{aligned}
\tag{2.23}
$$

For B,

$$
\begin{aligned}
B &= \int_{-\pi}^{\pi} \frac{e^{i\phi}}{(1+xe^{-i\phi})(1+xe^{i\phi})^2} f(\phi) d\phi \\
&\quad + \int_{-\pi}^{\pi} \frac{-(n+1)(-x)^n e^{i(n+1)\phi}\left(1+xe^{i\phi}\right)}{(1+xe^{-i\phi})(1+xe^{i\phi})^2} f(\phi) d\phi \\
&\quad - (-x)^{n+1} \int_{-\pi}^{\pi} \frac{f(\phi)}{(1+xe^{-i\phi})(1+xe^{i\phi})^2} \\
&\quad \cdot \left[-(n+1)(-x)^n\left(1+xe^{i\phi}\right) + e^{-in\phi}\left(1-(-x)^{n+1}e^{i(n+1)\phi}\right) + e^{i(n+2)\phi}\right] d\phi \\
&= B_1 + B_2 + O\left((-x)^{n+1}\left(|B_1|+|B_2|\right)\right).
\end{aligned}
$$

We will again show that the B_1 term dominates, resulting in the expression

$$B = B_1 + B_2 + O\left((-x)^{n+1} B_1\right). \tag{2.24}$$

Splitting B_1 into two integrals yields

$$\begin{aligned}
B_1 &= 2\int_{\pi-g(y)}^{\pi} \frac{\cos\phi + 1 - y}{\left(1 + 2(1-y)\cos\phi + (1-y)^2\right)^2} f(\phi)d\phi \\
&\quad + 2\int_0^{\pi-g(y)} \frac{\cos\phi + 1 - y}{\left(1 + 2(1-y)\cos\phi + (1-y)^2\right)^2} f(\phi)d\phi.
\end{aligned}$$

For the first integral we have

$$\begin{aligned}
&= 2\int_{\pi-g(y)}^{\pi} \frac{-y + \frac{(\phi-\pi)^2}{2} + O\left((\phi-\pi)^4\right)}{\left(1 + 2(1-y)\left(-1 + \frac{(\phi-\pi)^2}{2} + O\left((\phi-\pi)^4\right)\right) + (1-y)^2\right)^2} f(\phi)d\phi \\
&= 2\int_{\pi-g(y)}^{\pi} \frac{-y}{\left((\phi-\pi)^2 + y^2 - y(\phi-\pi)^2 + O\left((\phi-\pi)^4\right)\right)^2} f(\phi)d\phi + O\left(\frac{1}{y}\right) \\
&= 2\int_{\pi-g(y)}^{\pi} \frac{-y}{\left((\phi-\pi)^2 + y^2\right)^2} f(\phi)d\phi + O\left(\frac{1}{y}\right).
\end{aligned}$$

For $f \in C\left([-\pi,\pi]\right)$ this becomes

$$\begin{aligned}
&\sim -2\int_{\pi-g(y)}^{\pi} \frac{yf(\pi)}{\left((\phi-\pi)^2 + y^2\right)^2} d\phi \\
&= -\frac{f(\pi)}{y^2}\left[\frac{g(y)y}{g^2(y) + y^2} + \arctan\left(\frac{g(y)}{y}\right)\right] \\
&\sim -\frac{f(\pi)}{y^2}\arctan\left(\frac{g(y)}{y}\right),
\end{aligned}$$

while for $f \in C^1\left([-\pi,\pi]\right)$

$$\begin{aligned}
&= -2\int_{\pi-g(y)}^{\pi} \frac{yf(\pi) + yf'(\phi_0)(\phi-\pi)}{\left((\phi-\pi)^2 + y^2\right)^2} d\phi + O\left(\frac{1}{y}\right) \quad (\text{where } \phi_0 \in (\pi-\phi,\pi)) \\
&= -\frac{f(\pi)}{y^2}\left[\frac{(\phi-\pi)y}{(\phi-\pi)^2 + y^2} + \arctan\left(\frac{\phi-\pi}{y}\right)\right]\Bigg|_{\pi-g(y)}^{\pi} \\
&\quad + O\left(\int_{\pi-g(y)}^{\pi} \frac{-y(\phi-\pi)}{\left((\phi-\pi)^2 + y^2\right)^2} d\phi + \frac{1}{y}\right) \\
&= -\frac{f(\pi)}{y^2}\left[\frac{g(y)y}{g^2(y) + y^2} + \arctan\left(\frac{g(y)}{y}\right)\right] + O\left(\frac{1}{y}\right)
\end{aligned}$$

$$= -\frac{f(\pi)}{y^2}\arctan\left(\frac{g(y)}{y}\right) + O\left(\frac{1}{yg(y)}\right).$$

For the second integral in B_1 we have

$$\begin{aligned}
&\left|2\int_0^{\pi-g(y)} \frac{\cos\phi + x}{(1+2x\cos\phi + x^2)^2} f(\phi)d\phi\right| \\
&\leq 2\int_{\pi-(g(y))^{1/3}}^{\pi-g(y)} \frac{|\cos\phi + x|\, f(\phi)}{(1+2x\cos\phi+x^2)^2}d\phi + 2\int_0^{\pi-(g(y))^{1/3}} \frac{|\cos\phi + x|\, f(\phi)}{(1+2x\cos\phi+x^2)^2}d\phi \\
&\sim 2f(\pi)\int_{\pi-(g(y))^{1/3}}^{\pi-g(y)} \frac{\left|-y + \frac{(\phi-\pi)^2}{2}\right|}{((\phi-\pi)^2+y^2)}d\phi + 2\int_0^{\pi-(g(y))^{1/3}} \frac{|\cos\phi + x|\, f(\phi)}{(1+2x\cos\phi+x^2)^2}d\phi \\
&\leq 2f(\pi)\int_{\pi-(g(y))^{1/3}}^{\pi-g(y)} \frac{y+(\phi-\pi)^2}{(\phi-\pi)^4}d\phi + \int_0^{\pi-(g(y))^{1/3}} \frac{4M}{(1+2x\cos(g(y))^{1/3}+x^2)^2}d\phi \\
&\sim \frac{2f(\pi)y}{3}\left(\frac{1}{(g(y))^3} - \frac{1}{g(y)}\right) + 2f(\pi)\left(\frac{1}{g(y)} - \frac{1}{(g(y))^{1/3}}\right) + \frac{4\pi M}{(g(y))^{4/3}} \\
&= O\left(\frac{y}{(g(y))^3}\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
B_1 &\sim -\frac{f(\pi)}{y^2}\arctan\left(\frac{g(y)}{y}\right), \qquad \text{for } f \in C([-\pi,\pi]), \\
B_1 &= -\frac{f(\pi)}{y^2}\arctan\left(\frac{g(y)}{y}\right) + O\left(\frac{1}{yg(y)}\right), \qquad \text{for } f \in C^1([-\pi,\pi]).
\end{aligned}$$

Next, B_2 we have

$$\begin{aligned}
B_2 &= \left|\int_{-\pi}^{\pi} \frac{-(n+1)(-x)^n e^{i(n+1)\phi}(1+xe^{i\phi})}{(1+xe^{-i\phi})(1+xe^{i\phi})^2} f(\phi)d\phi\right| \\
&\leq \int_{-\pi}^{\pi} \frac{(n+1)x^n}{(1+xe^{-i\phi})(1+xe^{i\phi})} f(\phi)d\phi \\
&\sim c(n+1)\frac{(1-y)^n}{y} \\
&= O\left(\frac{1}{yg(y)}\right),
\end{aligned}$$

where the last line is given by (2.12). Thus,

$$\begin{aligned}
\left|(-x)^{n+1}(B_1+B_2)\right| &\sim c\frac{(1-y)^{n+1}}{y^2} \\
&< c(n+1)\frac{(1-y)^{n+1}}{y} \\
&= O\left(\frac{1}{yg(y)}\right).
\end{aligned}$$

It now follows that

$$
\begin{aligned}
B &\sim -\frac{f(\pi)}{y^2}\arctan\left(\frac{g(y)}{y}\right), \qquad \text{for } f \in C\left([-\pi,\pi]\right),\\
B &= -\frac{f(\pi)}{y^2}\arctan\left(\frac{g(y)}{y}\right) + O\left(\frac{1}{yg(y)}\right), \qquad \text{for } f \in C^1\left([-\pi,\pi]\right).
\end{aligned}
\tag{2.25}
$$

Considering C last,

$$
\begin{aligned}
C = &\int_{-\pi}^{\pi} \frac{1}{(1+xe^{-i\phi})^2(1+xe^{i\phi})^2} f(\phi)d\phi + \int_{-\pi}^{\pi} \frac{(n+1)^2(-x)^{2n}}{(1+xe^{-i\phi})(1+xe^{i\phi})} f(\phi)d\phi\\
&+ 2\int_{-\pi}^{\pi} \frac{-(n+1)(-x)^n e^{-in\phi}\left(1+(-x)^{n+1}e^{i(n+1)\phi}\right)}{(1+xe^{-i\phi})(1+xe^{i\phi})^2} f(\phi)d\phi\\
&+ (-x)^{n+1}\int_{-\pi}^{\pi} \frac{-\left(e^{-i(n+1)\phi}+e^{i(n+1)\phi}\right)+(-x)^{n+1}}{(1+xe^{-i\phi})^2(1+xe^{i\phi})^2} f(\phi)d\phi\\
&= C_1 + C_2 + C_3 + O\left((-x)^{n+1}C_1\right).
\end{aligned}
$$

Splitting C_1 into two integrals gives us

$$
\begin{aligned}
C_1 = &\, 2\int_{\pi-g(y)}^{\pi} \frac{1}{(1+2x\cos\phi+x^2)^2} f(\phi)d\phi\\
&+ 2\int_{0}^{\pi-g(y)} \frac{1}{(1+2x\cos\phi+x^2)^2} f(\phi)d\phi.
\end{aligned}
\tag{2.26}
$$

For the first term we have

$$
\begin{aligned}
&= 2\int_{\pi-g(y)}^{\pi} \frac{1}{\left(1+2(1-y)\left(-1+\frac{(\phi-\pi)^2}{2}+O\left((\phi-\pi)^4\right)\right)+(1-y)^2\right)^2} f(\phi)d\phi\\
&= 2\int_{\pi-g(y)}^{\pi} \frac{1}{\left((\phi-\pi)^2+y^2-y(\phi-\pi)^2+O\left((\phi-\pi)^4\right)\right)^2} f(\phi)d\phi\\
&= 2\int_{\pi-g(y)}^{\pi} \frac{f(\phi)}{\left((\phi-\pi)^2+y^2\right)^2} d\phi + O\left(\frac{1}{y^2}\right).
\end{aligned}
$$

For $f \in C\left([-\pi,\pi]\right)$ this yields

$$
\begin{aligned}
&\sim 2\int_{\pi-g(y)}^{\pi} \frac{f(\pi)}{\left((\phi-\pi)^2+y^2\right)^2} d\phi\\
&= \frac{f(\pi)}{y^3}\left[\frac{g(y)y}{g^2(y)+y^2} + \arctan\left(\frac{g(y)}{y}\right)\right]\\
&\sim \frac{f(\pi)}{y^3}\arctan\left(\frac{g(y)}{y}\right),
\end{aligned}
$$

while for $f \in C^1([-\pi, \pi])$

$$
\begin{aligned}
&= 2\int_{\pi-g(y)}^{\pi} \frac{f(\pi) + f'(\phi_0)(\phi-\pi)}{((\phi-\pi)^2+y^2)^2} d\phi + O\left(\frac{1}{y^2}\right) \\
&\quad (\text{where } \phi_0 \in (\pi-\phi, \pi)) \\
&= \frac{f(\pi)}{y^3}\left[\frac{(\phi-\pi)y}{(\phi-\pi)^2+y^2} + \arctan\left(\frac{\phi-\pi}{y}\right)\right]\Bigg|_{\pi-g(y)}^{\pi} \\
&\quad + O\left(\int_{\pi-g(y)}^{\pi} \frac{-(\phi-\pi)}{((\phi-\pi)^2+y^2)^2} d\phi + \frac{1}{y^2}\right) \\
&= \frac{f(\pi)}{y^3}\left[\frac{g(y)y}{g^2(y)+y^2} + \arctan\left(\frac{g(y)}{y}\right)\right] + O\left(\frac{1}{y^2}\right) \\
&= \frac{f(\pi)}{y^3}\arctan\left(\frac{g(y)}{y}\right) + O\left(\frac{1}{y^2 g(y)}\right).
\end{aligned}
$$

Looking at the second integral in C_1 we have

$$
\begin{aligned}
&= 2\int_{\pi-(g(y))^{1/3}}^{\pi-g(y)} \frac{f(\phi)}{(1+2x\cos\phi+x^2)^2} d\phi + 2\int_0^{\pi-(g(y))^{1/3}} \frac{f(\phi)}{(1+2x\cos\phi+x^2)^2} d\phi \\
&\sim 2f(\pi)\int_{\pi-(g(y))^{1/3}}^{\pi-g(y)} \frac{1}{(y^2+(\phi-\pi)^2)^2} d\phi + 2\int_0^{\pi-(g(y))^{1/3}} \frac{f(\phi)}{(1+2x\cos\phi+x^2)^2} d\phi \\
&\le 2f(\pi)\int_{\pi-(g(y))^{1/3}}^{\pi-g(y)} \frac{1}{(\phi-\pi)^4} d\phi + 2\int_0^{\pi-(g(y))^{1/3}} \frac{M}{(1+2x\cos((g(y))^{1/3})+x^2)^2} d\phi \\
&\sim 2f(\pi)\left(\frac{1}{(g(y))^3} - \frac{1}{g(y)}\right) + \frac{2M\pi}{(g(y))^{4/3}} \\
&= O\left(\frac{1}{(g(y))^3}\right).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&C_1 \sim \frac{f(\pi)}{y^3}\arctan\left(\frac{g(y)}{y}\right), \qquad \text{for } f \in C([-\pi, \pi]), \\
&C_1 = \frac{f(\pi)}{y^3}\arctan\left(\frac{g(y)}{y}\right) + O\left(\frac{1}{y^2 g(y)}\right), \qquad \text{for } f \in C^1([-\pi, \pi]).
\end{aligned}
\tag{2.27}
$$

For C_2 we have,

$$
C_2 = \int_{-\pi}^{\pi} \frac{(n+1)^2 x^{2n}(1+xe^{i\phi})(1+xe^{-i\phi})}{(1+xe^{-i\phi})^2(1+xe^{i\phi})^2} f(\phi) d\phi
$$

$$\sim c(n+1)^2 \frac{(1-y)^{2n}}{y},$$

while for C_3,

$$\begin{aligned} |C_3| &\le c \int_{-\pi}^{\pi} \left| \frac{-(n+1)(-x)^n e^{-in\phi}}{(1+xe^{-i\phi})(1+xe^{i\phi})^2} f(\phi) \right| d\phi \\ &\le c (C_1 C_2)^{1/2} \quad \text{(by Cauchy-Schwarz)} \\ &\sim c \frac{(n+1)(1-y)^n}{y^2}. \end{aligned}$$

Also,

$$|(-x)^{n+1} C_1| \sim c \frac{(1-y)^{n+1}}{y^3} \le \frac{(n+1)(1-y)^n}{y^2}.$$

Applying (2.16), we can now conclude that

$$\begin{aligned} C &\sim \frac{f(\pi)}{y^3} \arctan\left(\frac{g(y)}{y}\right), \qquad \text{for } f \in C\left([-\pi,\pi]\right), \\ C &= \frac{f(\pi)}{y^3} \arctan\left(\frac{g(y)}{y}\right) + O\left(\frac{1}{y^2 g(y)}\right), \qquad \text{for } f \in C^1\left([-\pi,\pi]\right). \end{aligned} \tag{2.28}$$

Combining (2.23), (2.25), and (2.28), for $f \in C\left([-\pi,\pi]\right)$

$$\begin{aligned} \frac{\sqrt{AC - B^2}}{A} &\sim \sqrt{\frac{\pi f(\pi)}{y} \cdot \frac{\pi f(\pi)}{2y^3} - \left(\frac{\pi f(\pi)}{2y^2}\right)^2} \left(\frac{-\pi f(\pi)}{y}\right)^{-1} \\ &= \frac{1}{2y}. \end{aligned}$$

For $f \in C^1\left([-\pi,\pi]\right)$ we now have

$$\begin{aligned} (2.29)\quad & AC - B^2 = \\ & \left[\frac{2f(\pi)}{y} \arctan\left(\frac{g(y)}{y}\right) + O\left(\frac{1}{g(y)}\right)\right] \left[\frac{f(\pi)}{y^3} \arctan\left(\frac{g(y)}{y}\right) + O\left(\frac{1}{y^2 g(y)}\right)\right] \\ & \qquad - \left[\frac{-f(\pi)}{y^2} \arctan\left(\frac{g(y)}{y}\right) + O\left(\frac{1}{y g(y)}\right)\right]^2 \\ & \qquad = \frac{f^2(\pi)}{y^4} \left[\arctan\left(\frac{g(y)}{y}\right)\right]^2 + O\left(\frac{1}{y^3 g(y)}\right), \end{aligned}$$

and

$$\frac{\sqrt{AC-B^2}}{A} = \frac{\sqrt{\frac{f^2(\pi)}{y^4}\left[\arctan\left(\frac{g(y)}{y}\right)\right]^2 + O\left(\frac{1}{y^3 g(y)}\right)}}{\frac{2f(\pi)}{y}\arctan\left(\frac{g(y)}{y}\right) + O\left(\frac{1}{g(y)}\right)}$$

$$(2.30) \qquad = \frac{\sqrt{\frac{f^2(\pi)}{y^4}\left[\arctan\left(\frac{g(y)}{y}\right)\right]^2 + O\left(\frac{1}{y^3 g(y)}\right)}}{\frac{2f(\pi)}{y}\arctan\left(\frac{g(y)}{y}\right)} + O\left(\frac{1}{g(y)}\right)$$

$$= \frac{1}{2y} + O\left(\frac{1}{g(y)}\right).$$

Plugging these into (2.1) gives us

$$\mathrm{E}\left[N\left(-1+\frac{\log\log n}{n}, -1+\frac{1}{\log n}\right)\right] = \frac{1}{\pi}\int_{-1+\frac{\log\log n}{n}}^{-1+\frac{1}{\log n}} \frac{\sqrt{AC-B^2}}{A}dx$$

$$(2.31) \qquad \sim \frac{1}{\pi}\int_{\frac{\log\log n}{n}}^{\frac{1}{\log n}} \frac{1}{2y}dy$$

$$= \frac{1}{2\pi}\log y\Big|_{\frac{\log\log n}{n}}^{\frac{1}{\log n}}$$

$$\sim \frac{1}{2\pi}\log n,$$

for $f \in C([-\pi,\pi])$, and

$$\mathrm{E}\left[N\left(-1+\frac{\log\log n}{n}, -1+\frac{1}{\log n}\right)\right] = \frac{1}{\pi}\int_{-1+\frac{\log\log n}{n}}^{-1+\frac{1}{\log n}} \frac{\sqrt{AC-B^2}}{A}dx$$

$$(2.32) \qquad = \frac{1}{\pi}\int_{\frac{\log\log n}{n}}^{\frac{1}{\log n}} \left[\frac{1}{2y} + O\left(\frac{1}{g(y)}\right)\right]dy$$

$$= \frac{1}{2\pi}\log y\Big|_{\frac{\log\log n}{n}}^{\frac{1}{\log n}} + O(\log\log n)$$

$$= \frac{1}{2\pi}\log n + O(\log\log n),$$

for $f \in C^1([-\pi,\pi])$.

□

We are now able to prove Theorem 2.1.1.

Proof of Theorem 2.1.1. Combining the results of Lemmas 2.2.1 and 2.3.1, we have

$$\mathrm{E}[N(-1,1)] \sim \frac{1}{\pi}\log n, \qquad \text{for } f \in C([-\pi,\pi]),$$

$$E[N(-1,1)] = \frac{1}{\pi}\log n + O(\log\log n), \qquad \text{for } f \in C([-\pi,\pi]).$$

From our discussion in the comments preceding section 1 we know that

$$E[N(-\infty,\infty)] = 2E[N(-1,1)].$$

It then follows that

$$E[N(-\infty,\infty)] \sim \frac{2}{\pi}\log n, \qquad \text{for } f \in C([-\pi,\pi]),$$
$$E[N(-\infty,\infty)] = \frac{2}{\pi}\log n + O(\log\log n), \qquad \text{for } f \in C([-\pi,\pi]),$$

as claimed. □

2.4 Conclusions

We have shown that, for the expected number of real zeros, behavior similar to the independent case holds for a wide class of covariance functions. However, noting the change in behavior when the covariance is constant, it would be of interest to see exactly where this change in behavior takes place, and what would happen for covariance functions with slower rates of decay. For example, if $\Gamma(k) = \frac{1}{|k|+1}$, the techniques developed no longer apply. Thus, a new approach would have to be taken to answer these questions.

Chapter 3

K-level Crossings

3.1 Introduction

For the random polynomial given by (1), consider the problem of computing the expected number of real zeros for the equation $P_n(x) = K$, where K is a given constant. These are known as the K-level crossings of $P_n(x)$. For standard normal coefficients, Farahmand considered this for two separate cases [11, 12]. The first assumes the coefficients are independent, while the second deals with dependent coefficients with a constant covariance ρ, where $\rho \in (0, 1)$. In this chapter we will study further the case of dependent coefficients.

Our will results will cover two different assumptions on K, similar to the ones considered by Farahmand. The first assumes that K is bounded. If we require only that the spectral density is continuous and positive, we will be able to show that the expected number of level crossings will behave asymptotically like $\frac{2}{\pi} \log n$ as $n \to \infty$. For the second situation, we will let K grow along with n. Under the assumptions that $K = o\left(\sqrt{\frac{n}{\log \log n}}\right)$ and that the spectral density is positive and in $C^1\left([-\pi, \pi]\right)$, we will be able to show that the expected number of crossings in the interval $(-1, 1)$ is reduced. Recalling that $N_K(\alpha, \beta)$ is the number of K-level crossings of $P_n(x)$ in the interval (α, β), these results are formulated as follows.

Theorem 3.1.1. *Assume that the spectral density exists and is strictly positive.*

(i) For K bounded and $f(\phi) \in C([-\pi, \pi])$ we have

$$\mathrm{E}\left[N_K\left(-1,1\right)\right] = \mathrm{E}\left[N_K\left(-\infty,-1\right) + N_K\left(1,\infty\right)\right] \sim \frac{1}{\pi}\log n.$$

(ii) For $K = o\left(\sqrt{\frac{n}{\log\log n}}\right)$ and $f(\phi) \in C^1([-\pi, \pi])$ we have

$$\mathrm{E}\left[N_K\left(-1,1\right)\right] = \frac{1}{\pi}\log\frac{n}{K^2} + O\left(\log\log n\right),$$
$$\mathrm{E}\left[N_K\left(-\infty,-1\right) + N_K\left(1,\infty\right)\right] = \frac{1}{\pi}\log n + O\left(\log\log n\right).$$

Recalling the Kac-Rice formula (1.6.1),

$$\begin{aligned}
\mathrm{E}\left[N_K\left(\alpha,\beta\right)\right] &= \frac{1}{\pi}\int_\alpha^\beta \frac{\sqrt{AC-B^2}}{A}\exp\left(-\frac{K^2C}{2\left(AC-B^2\right)}\right)dx \\
&\quad + \frac{1}{\pi}\int_\alpha^\beta \frac{\sqrt{2}|BK|}{A^{3/2}}\exp\left(-\frac{K^2}{2A}\right)\mathrm{erf}\left(\frac{|-BK|}{\sqrt{2A\left(AC-B^2\right)}}\right)dx \qquad (3.1)\\
&= \int_\alpha^\beta F_1 dx + \int_\alpha^\beta F_2 dx,
\end{aligned}$$

where

$$A(x) = \mathrm{E}[P_n^2(x)] = \sum_{k=0}^{n}\sum_{j=0}^{n}\Gamma(k-j)x^{k+j},$$
$$B(x) = \mathrm{E}[P_n(x)P_n'(x)] = \sum_{k=0}^{n}\sum_{j=0}^{n}\Gamma(k-j)kx^{k+j-1},$$
$$C(x) = \mathrm{E}[(P_n'(x))^2] = \sum_{k=0}^{n}\sum_{j=0}^{n}\Gamma(k-j)kjx^{k+j-2}.$$

Applying (1.2) gives us

$$A = \int_{-\pi}^{\pi}\sum_{k=0}^{n}\sum_{j=0}^{n}e^{-i(k-j)\phi}x^{k+j}f(\phi)d\phi,$$
$$B = \int_{-\pi}^{\pi}\sum_{k=0}^{n}\sum_{j=0}^{n}e^{-i(k-j)\phi}kx^{k+j-1}f(\phi)d\phi,$$
$$C = \int_{-\pi}^{\pi}\sum_{k=0}^{n}\sum_{j=0}^{n}e^{-i(k-j)\phi}kjx^{k+j-2}f(\phi)d\phi.$$

Recall that from (2.2), (2.3), and (2.4) we have the expressions

$$
\begin{aligned}
A &= \int_{-\pi}^{\pi} H(x,x) f(\phi) d\phi \\
&= \int_{-\pi}^{\pi} \frac{1 - x^{n+1} e^{-i(n+1)\phi}}{1 - x e^{-i\phi}} \cdot \frac{1 - x^{n+1} e^{i(n+1)\phi}}{1 - x e^{i\phi}} f(\phi) d\phi,
\end{aligned}
$$

$$
\begin{aligned}
B &= \int_{-\pi}^{\pi} \left[\frac{\partial H(x,y)}{\partial y} \right]_{y=x} f(\phi) d\phi \\
&= \int_{-\pi}^{\pi} \left(\frac{1 - x^{n+1} e^{-i(n+1)\phi}}{1 - x e^{-i\phi}} \right) \\
&\quad \cdot \left(\frac{-(n+1) x^n e^{i(n+1)\phi} (1 - x e^{i\phi}) - (1 - x^{n+1} e^{i(n+1)\phi})(-e^{i\phi})}{(1 - x e^{i\phi})^2} \right) f(\phi) d\phi,
\end{aligned}
$$

and

$$
\begin{aligned}
C &= \int_{-\pi}^{\pi} \left[\frac{\partial^2 H(x,y)}{\partial x \partial y} \right]_{y=x} f(\phi) d\phi \\
&= \int_{-\pi}^{\pi} \left(\frac{-(n+1) x^n e^{-i(n+1)\phi} (1 - x e^{-i\phi}) - (1 - x^{n+1} e^{-i(n+1)\phi})(-e^{-i\phi})}{(1 - x e^{-i\phi})^2} \right) \\
&\quad \cdot \left(\frac{-(n+1) x^n e^{i(n+1)\phi} (1 - x e^{i\phi}) - (1 - x^{n+1} e^{i(n+1)\phi})(-e^{i\phi})}{(1 - x e^{i\phi})^2} \right) f(\phi) d\phi.
\end{aligned}
$$

3.2 Expected Number of Level Crossings on $(-1,1)$

Our first step will be to show that the contribution from the integral of F_2 is negligible.

Lemma 3.2.1. *For $f(\phi)$ continuous and positive we have*

$$\int_{-1}^{1} F_2 dx = o(\log \log n).$$

Proof. Since $f(\phi)$ is a continuous, positive function, we can find constants $c_1, c_2 > 0$ such that $\frac{c_1}{2\pi} > f(\phi) > \frac{c_2}{2\pi}$ for any $\phi \in [-\pi, \pi]$. Now, for the interval $(-1 + \frac{\log\log n}{n}, 1 - \frac{\log\log n}{n})$ we have

$$A \sim \int_{-\pi}^{\pi} \frac{1}{(1 - xe^{-i\phi})(1 - xe^{i\phi})} f(\phi) d\phi,$$

from which we can then derive the lower bound

$$\frac{c_2\,(1 - x^{2n+2})}{1 - x^2} \leq \frac{c_2}{1 - x^2} = \frac{c_2}{2\pi} \int_{-\pi}^{\pi} \frac{1}{(1 - xe^{-i\phi})(1 - xe^{i\phi})} d\phi \leq A. \tag{3.2}$$

Using the fact that $f \equiv \frac{1}{2\pi}$ in the independent case, we can derive an upper bound as well, where

$$\begin{aligned} A &\leq \frac{c_1}{2\pi} \int_{-\pi}^{\pi} \frac{\left(1 - x^{n+1} e^{-i(n+1)\phi}\right)\left(1 - x^{n+1} e^{i(n+1)\phi}\right)}{(1 - xe^{-i\phi})(1 - xe^{i\phi})} d\phi \\ &= c_1 \frac{1 - x^{2n+2}}{1 - x^2} \\ &\leq \frac{c_1}{1 - x^2}. \end{aligned} \tag{3.3}$$

Notice that this upper bound holds on the entire interval $(0, 1)$. Next, from our work in Chapter 2 we know that

$$|B| \sim \int_{-\pi}^{\pi} \left| \frac{e^{i\phi}}{(1 - xe^{-i\phi})(1 - xe^{i\phi})^2} \right| f(\phi) d\phi,$$

which implies

$$\begin{aligned} |B| &\leq \frac{1}{1 - |x|} \int_{-\pi}^{\pi} \frac{1}{(1 - xe^{-i\phi})(1 - xe^{i\phi})} f(\phi) d\phi \\ &\sim \frac{1}{1 - |x|} A. \end{aligned}$$

It follows that

$$\begin{aligned} \frac{|B|}{A^{3/2}} &\leq \frac{1}{1 - |x|} \left(\frac{1 - x^2}{c_2} \right)^{1/2} \\ &\leq \sqrt{\frac{2}{c_2}} \frac{1}{(1 - |x|)^{1/2}}, \end{aligned}$$

and

$$\exp\left(\frac{-K^2}{2A} \right) \leq \frac{1}{1 + \frac{K^2}{2A}}$$

$$\leq \frac{1}{1+\frac{K^2(1-x^2)}{2c_1}}$$
$$\leq \frac{1}{1+\frac{K^2(1-|x|)}{2c_1}}.$$

Thus,

$$\begin{aligned}
\int_{-1+\frac{\log\log n}{n}}^{1-\frac{\log\log n}{n}} F_2 dx &\leq \sqrt{\frac{2}{c_2}} \int_{-1+\frac{\log\log n}{n}}^{1-\frac{\log\log n}{n}} \frac{|K|(1-|x|)^{-1/2}}{1+\frac{K^2(1-|x|)}{2c_1}} dx \\
&= 2\sqrt{\frac{2}{c_2}} \int_{0}^{1-\frac{\log\log n}{n}} \frac{|K|(1-x)^{-1/2}}{1+\frac{K^2(1-x)}{2c_1}} dx \\
&= -2\sqrt{2c_1} \arctan\left(\frac{K\sqrt{1-x}}{\sqrt{2c_1}}\right)\Bigg|_0^{1-\frac{\log\log n}{n}} \\
&= O(1).
\end{aligned} \tag{3.4}$$

Next, for $x \in (-1, -1+\frac{\log\log n}{n}) \cup (1-\frac{\log\log n}{n}, 1)$,

$$\begin{aligned}
|B| &\leq \frac{n}{|x|} \sum_{k=0}^{n} \sum_{j=0}^{n} \Gamma(k-j)|x|^{k+j} \\
&\leq \frac{nc_1}{|x|} \sum_{k=0}^{n} x^{2k},
\end{aligned}$$

by (3.3). We also have

$$\begin{aligned}
A &\geq \frac{c_2}{2\pi} \int_{-\pi}^{\pi} \frac{\left(1-x^{n+1}e^{-i(n+1)\phi}\right)\left(1-x^{n+1}e^{i(n+1)\phi}\right)}{(1-xe^{-i\phi})(1-xe^{i\phi})} d\phi \\
&= c_2 \sum_{k=0}^{n} x^{2k},
\end{aligned}$$

from which it then follows that

$$\begin{aligned}
\frac{|B|}{A^{3/2}} &\leq nc\left(\sum_{k=0}^{n} x^{2k}\right)^{-1/2} \\
&\leq nc\left(\sum_{k=0}^{n}\left(1-\frac{\log\log n}{n}\right)^{2k}\right)^{-1/2} \\
&\sim nc\left(2\frac{\log\log n}{n}\right)^{1/2} \\
&\sim c\,(n\log\log n)^{1/2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{\sqrt{2}}{\pi}\int_{1-\frac{\log\log n}{n}}^{1} F_2 &\leq \frac{\sqrt{2}}{\pi}\int_{1-\frac{\log\log n}{n}}^{1} \frac{|KB|}{A^{3/2}} \\
&\leq \pi\int_{1-\frac{\log\log n}{n}}^{1} c|K|\left(n\log\log n\right)^{1/2} \\
&= c|K|\frac{(\log\log n)^{3/2}}{n^{1/2}} \\
&= o\left(\log\log n\right).
\end{aligned}$$

Similarly,

$$\frac{\sqrt{2}}{\pi}\int_{-1}^{-1+\frac{\log\log n}{n}} F_2 = o(\log\log n),$$

which proves the claim. □

We will next show that the expected number of zeros on the intervals $(0, 1-\frac{1}{\log n})$, $(1-\frac{\log\log n}{n}, 1)$, $(-1+\frac{1}{\log n}, 0)$ and $(-1, -1+\frac{\log\log n}{n})$ is negligible.

Lemma 3.2.2. *Assume $f(\phi)$ is continuous and positive. For the intervals $(-1, -1+\frac{\log\log n}{n})$, $(-1+\frac{1}{\log n}, 0)$, $(0, 1-\frac{1}{\log n})$, and $(1-\frac{\log\log n}{n}, 1)$, the expected number of zeros is $O(\log\log n)$.*

Proof. To start, we note that since the quantity $\frac{K^2C}{AC-B^2}$ is never negative, the inequality

$$\exp\left(-\frac{K^2C}{2\left(AC-B^2\right)}\right) \leq 1$$

holds in general. It follows that

$$\int_{\alpha}^{\beta} F_1 dx \leq \frac{1}{\pi}\int_{\alpha}^{\beta} \frac{\sqrt{AC-B^2}}{A} dx. \tag{3.5}$$

Applying Lemma 3.2.1 from above, along with Lemma 2.2.1, we then have

$$\mathrm{E}\left[N\left(-1+\frac{1}{\log n}, 1-\frac{1}{\log n}\right)\right] = O\left(\log\log n\right),$$

and

$$\mathrm{E}\left[N\left(-1, -1+\frac{\log\log n}{n}\right)\right] = \mathrm{E}\left[N\left(1-\frac{\log\log n}{n}, 1\right)\right] = O\left(\log\log n\right).$$

□

In what remains of this section we will mainly be concerned with computing F_1 on the intervals $(-1+\frac{\log\log n}{n}, -1+\frac{1}{\log n})$ and $(1-\frac{1}{\log n}, 1-\frac{\log\log n}{n})$. Recall that $g(y) = y\frac{\log n}{\log\log n}$. Starting with $x = 1-y \in (1-\frac{1}{\log n}, 1-\frac{\log\log n}{n})$, from our work in Chapter 2 we have the equations

$$\begin{aligned} A &\sim \frac{2f(0)}{y}\arctan\left(\frac{g(y)}{y}\right), \\ B &\sim \frac{f(0)}{y^2}\arctan\left(\frac{g(y)}{y}\right), \\ C &\sim \frac{f(0)}{y^3}\arctan\left(\frac{g(y)}{y}\right), \end{aligned} \tag{3.6}$$

for $f(\phi) \in C([-\pi,\pi])$, and

$$\begin{aligned} A &= \frac{2f(0)}{y}\arctan\left(\frac{g(y)}{y}\right) + O\left(\frac{1}{g(y)}\right), \\ B &= \frac{f(0)}{y^2}\arctan\left(\frac{g(y)}{y}\right) + O\left(\frac{1}{yg(y)}\right), \\ C &= \frac{f(0)}{y^3}\arctan\left(\frac{g(y)}{y}\right) + O\left(\frac{1}{y^2g(y)}\right), \end{aligned} \tag{3.7}$$

for $f(\phi) \in C^1([-\pi,\pi])$. Also from Chapter 2, we have the expressions

$$\begin{aligned} AC - B^2 &\sim \frac{f^2(0)}{y^4}\left[\arctan\left(\frac{g(y)}{y}\right)\right]^2, \\ \frac{\sqrt{AC-B^2}}{A} &\sim \frac{1}{2y}, \end{aligned} \tag{3.8}$$

for $f(\phi) \in C([-\pi,\pi])$, and

$$\begin{aligned} AC - B^2 &= \frac{f^2(0)}{y^4}\left[\arctan\left(\frac{g(y)}{y}\right)\right]^2 + O\left(\frac{1}{y^3g(y)}\right), \\ \frac{\sqrt{AC-B^2}}{A} &= \frac{1}{2y} + O\left(\frac{1}{g(y)}\right), \end{aligned} \tag{3.9}$$

for $f(\phi) \in C^1([-\pi,\pi])$.

We will first handle the simpler case when $f(\phi) \in C([-\pi,\pi])$ and K is bounded. From (3.6) and (3.8) we have

$$\frac{CK^2}{2(AC-B^2)} \sim \frac{K^2 y}{2f(0)\arctan\left(\frac{g(y)}{y}\right)}.$$

It follows that

$$
\begin{aligned}
&\frac{1}{\pi}\int_{\frac{\log\log n}{n}}^{\frac{1}{\log n}} \frac{1}{2y}\exp\left(\frac{-K^2 y}{2f(0)\arctan\left(\frac{g(y)}{y}\right)}\right)dy\\
&\sim \frac{1}{\pi}\int_{\frac{\log\log n}{n}}^{\frac{1}{\log n}} \frac{1}{2y}\left(1-\frac{K^2 y}{2f(0)\arctan\left(\frac{g(y)}{y}\right)}\right)dy\\
&\sim \frac{1}{\pi}\int_{\frac{\log\log n}{n}}^{\frac{1}{\log n}} \frac{1}{2y}\\
&\sim \frac{1}{2\pi}\log n.
\end{aligned}
$$

Next, let $f(\phi)\in C^1([-\pi,\pi])$ and $K=o\left(\sqrt{\frac{n}{\log\log n}}\right)$. Applying (3.7) and (3.9) yields

$$
\begin{aligned}
\frac{CK^2}{2(AC-B^2)} &= \frac{K^2}{2}\left[\frac{f(0)}{y^3}\arctan\left(\frac{g(y)}{y}\right)+O\left(\frac{1}{y^2 g(y)}\right)\right]\\
&\quad\cdot\left[\frac{f^2(0)}{y^4}\left[\arctan\left(\frac{g(y)}{y}\right)\right]^2+O\left(\frac{1}{y^3 g(y)}\right)\right]^{-1}\\
&= \frac{K^2 y}{2f(0)\arctan\left(\frac{g(y)}{y}\right)}+O\left(\frac{K^2y^2}{g(y)}\right).
\end{aligned}
$$

Now, we can choose positive constants a_1 and a_2 such that for large enough n,

$$
\begin{aligned}
\frac{a_1K^2 y}{2f(0)\arctan\left(\frac{g(y)}{y}\right)} &\le \frac{K^2 y}{2f(0)\arctan\left(\frac{g(y)}{y}\right)}+O\left(\frac{K^2y^2}{g(y)}\right)\\
&\le \frac{a_2K^2 y}{2f(0)\arctan\left(\frac{g(y)}{y}\right)},
\end{aligned}
$$

which then yields

$$
\begin{aligned}
(3.10)\quad &\left[\frac{1}{2y}+O\left(\frac{1}{g(y)}\right)\right]\exp\left(\frac{-a_2K^2 y}{2f(0)\arctan\left(\frac{g(y)}{y}\right)}\right)\\
&\le F_1 \le \left[\frac{1}{2y}+O\left(\frac{1}{g(y)}\right)\right]\exp\left(\frac{-a_1K^2 y}{2f(0)\arctan\left(\frac{g(y)}{y}\right)}\right).
\end{aligned}
$$

For $i = 1, 2$ we have

$$\left[\frac{1}{2y} + O\left(\frac{1}{g(y)}\right)\right] \exp\left(\frac{-a_i K^2 y}{2f(0)\arctan\left(\frac{g(y)}{y}\right)}\right)$$
$$= \frac{1}{2y}\exp\left(\frac{-a_i K^2 y}{2f(0)\arctan\left(\frac{g(y)}{y}\right)}\right) + O\left(\frac{1}{g(y)}\right).$$

Thus, using an argument similar to the one in [11],

$$\frac{1}{\pi}\int_{\frac{\log\log n}{n}}^{\frac{1}{\log n}} \left[\frac{1}{2y}\exp\left(\frac{-a_i K^2 y}{2f(0)\arctan\left(\frac{g(y)}{y}\right)}\right) + O\left(\frac{1}{g(y)}\right)\right] dy$$
$$= \frac{1}{\pi}\int_{\frac{\log\log n}{n}}^{\frac{1}{\log n}} \frac{1}{2y}\exp\left(-cK^2 y\right) dy + O(\log\log n)$$
$$\left(\text{where } c = a_i\left[2f(0)\arctan\left(\frac{g(y)}{y}\right)\right]^{-1}\right)$$
$$= \frac{1}{2\pi}\left[\log\left(cK^2\frac{1}{\log n}\right) - \log\left(cK^2\frac{\log\log n}{n}\right)\right]$$
$$+ \frac{1}{2\pi}\int_0^{cK^2\frac{\log\log n}{n}} \frac{1-e^{-t}}{t}dt - \frac{1}{2\pi}\int_0^{cK^2\frac{1}{\log n}} \frac{1-e^{-t}}{t}dt$$
$$= \frac{1}{2\pi}\log n + \frac{1}{2\pi}\int_0^{cK^2\frac{\log\log n}{n}} \frac{1-e^{-t}}{t}dt - \frac{1}{2\pi}\int_0^{cK^2\frac{1}{\log n}} \frac{1-e^{-t}}{t}dt + O(\log\log n).$$

Since we are assuming that $K^2\frac{\log\log n}{n} \to 0$ as $n \to \infty$, the first integral is $o(1)$. For the second we have

$$= -\frac{1}{2\pi}\int_1^{cK^2\frac{1}{\log n}} \frac{1-e^{-t}}{t}dt - \frac{1}{2\pi}\int_0^1 \frac{1-e^{-t}}{t}dt$$
$$= -\frac{1}{2\pi}\int_1^{cK^2\frac{1}{\log n}} \frac{1}{t}dt + \frac{1}{2\pi}\int_1^{cK^2\frac{1}{\log n}} \frac{e^{-t}}{t}dt + O(1)$$
$$= -\frac{1}{2\pi}\log K^2 + O(\log\log n).$$

From (3.10) it then follows that

$$\frac{1}{\pi}\int_{1-\frac{1}{\log n}}^{1-\frac{\log\log n}{n}} F_1 = \frac{1}{2\pi}\log\left(\frac{n}{K^2}\right) + O(\log\log n). \tag{3.11}$$

To handle the interval from $(-1+\frac{\log\log n}{n}, -1+\frac{1}{\log n})$ we will substitute in $-x = -1+y$, where $x \in (1-\frac{1}{\log n}, 1-\frac{\log\log n}{n})$. Then

$$A = \int_{-\pi}^{\pi} \frac{1-(-x)^{n+1}e^{-i(n+1)\phi}}{1+xe^{-i\phi}} \cdot \frac{1-(-x)^{n+1}e^{i(n+1)\phi}}{1+xe^{i\phi}} f(\phi)d\phi,$$

$$B = \int_{-\pi}^{\pi} \left(\frac{1 - (-x)^{n+1} e^{-i(n+1)\phi}}{1 + xe^{-i\phi}} \right)$$
$$\cdot \left(\frac{-(n+1)(-x)^n e^{i(n+1)\phi}(1 + xe^{i\phi}) - (1 - (-x)^{n+1} e^{i(n+1)\phi})(-e^{i\phi})}{(1 + xe^{i\phi})^2} \right) f(\phi) d\phi,$$

and

$$C = \int_{-\pi}^{\pi} \left(\frac{-(n+1)(-x)^n e^{-i(n+1)\phi}(1 + xe^{-i\phi}) - (1 - (-x)^{n+1} e^{-i(n+1)\phi})(-e^{-i\phi})}{(1 + xe^{-i\phi})^2} \right)$$
$$\cdot \left(\frac{-(n+1)(-x)^n e^{i(n+1)\phi}(1 + xe^{i\phi}) - (1 - (-x)^{n+1} e^{i(n+1)\phi})(-e^{i\phi})}{(1 + xe^{i\phi})^2} \right) f(\phi) d\phi.$$

Referring to Chapter 2 once more,

$$\begin{aligned} A &\sim \frac{2f(\pi)}{y} \arctan\left(\frac{g(y)}{y}\right), \\ B &\sim -\frac{f(\pi)}{y^2} \arctan\left(\frac{g(y)}{y}\right), \\ C &\sim \frac{f(\pi)}{y^3} \arctan\left(\frac{g(y)}{y}\right), \end{aligned} \tag{3.12}$$

for $f(\phi) \in C([-\pi, \pi])$, and

$$\begin{aligned} A &= \frac{2f(\pi)}{y} \arctan\left(\frac{g(y)}{y}\right) + O\left(\frac{1}{g(y)}\right), \\ B &= -\frac{f(\pi)}{y^2} \arctan\left(\frac{g(y)}{y}\right) + O\left(\frac{1}{yg(y)}\right), \\ C &= \frac{f(\pi)}{y^3} \arctan\left(\frac{g(y)}{y}\right) + O\left(\frac{1}{y^2 g(y)}\right), \end{aligned} \tag{3.13}$$

for $f(\phi) \in C^1([-\pi, \pi])$. We now have the expressions

$$\begin{aligned} AC - B^2 &\sim \frac{f^2(\pi)}{y^4} \left[\arctan\left(\frac{g(y)}{y}\right)\right]^2, \\ \frac{\sqrt{AC - B^2}}{A} &\sim \frac{1}{2y}, \end{aligned} \tag{3.14}$$

for $f(\phi) \in C([-\pi, \pi])$, and

$$\begin{aligned} AC - B^2 &= \frac{f^2(\pi)}{y^4} \left[\arctan\left(\frac{g(y)}{y}\right)\right]^2 + O\left(\frac{1}{y^3 g(y)}\right), \\ \frac{\sqrt{AC - B^2}}{A} &= \frac{1}{2y} + O\left(\frac{1}{g(y)}\right), \end{aligned} \tag{3.15}$$

for $f(\phi) \in C^1([-\pi,\pi])$.

We will again start with the simpler case when $f(\phi) \in C([-\pi,\pi])$ and K is bounded. By (3.12) and (3.14),

$$\frac{CK^2}{2(AC-B^2)} \sim \frac{K^2 y}{2f(\pi)\arctan\left(\frac{g(y)}{y}\right)},$$

from which it then follows that

$$\begin{aligned}
&\frac{1}{\pi}\int_{\frac{\log\log n}{n}}^{\frac{1}{\log n}} \frac{1}{2y}\exp\left(\frac{-K^2 y}{2f(\pi)\arctan\left(\frac{g(y)}{y}\right)}\right)dy \\
&\sim \frac{1}{\pi}\int_{\frac{\log\log n}{n}}^{\frac{1}{\log n}} \frac{1}{2y}\left(1-\frac{K^2 y}{2f(\pi)\arctan\left(\frac{g(y)}{y}\right)}\right)dy \\
&\sim \frac{1}{\pi}\int_{\frac{\log\log n}{n}}^{\frac{1}{\log n}} \frac{1}{2y} \\
&\sim \frac{1}{2\pi}\log n.
\end{aligned}$$

Next, we will assume $f(\phi) \in C^1([-\pi,\pi])$ and $K = o\left(\sqrt{\frac{n}{\log\log n}}\right)$. Using (3.13) and (3.15) gives us

$$\begin{aligned}
\frac{CK^2}{2(AC-B^2)} &= \frac{K^2}{2}\left[\frac{f(\pi)}{y^3}\arctan\left(\frac{g(y)}{y}\right)+O\left(\frac{1}{y^2 g(y)}\right)\right] \\
&\quad\cdot\left[\frac{f^2(\pi)}{y^4}\left[\arctan\left(\frac{g(y)}{y}\right)\right]^2+O\left(\frac{1}{y^3 g(y)}\right)\right]^{-1} \\
&= \frac{K^2 y}{2f(\pi)\arctan\left(\frac{g(y)}{y}\right)}+O\left(\frac{K^2 y^2}{g(y)}\right).
\end{aligned}$$

As before, we can choose positive constants a_1 and a_2 such that

$$\begin{aligned}
\frac{a_1 K^2 y}{2f(\pi)\arctan\left(\frac{g(y)}{y}\right)} &\le \frac{K^2 y}{2f(\pi)\arctan\left(\frac{g(y)}{y}\right)}+O\left(\frac{K^2 y^2}{g(y)}\right) \\
&\le \frac{a_2 K^2 y}{2f(\pi)\arctan\left(\frac{g(y)}{y}\right)},
\end{aligned}$$

which then yields

$$(3.16)\quad \left[\frac{1}{2y}+O\left(\frac{1}{g(y)}\right)\right]\exp\left(\frac{-a_2K^2y}{2f(\pi)\arctan\left(\frac{g(y)}{y}\right)}\right)$$
$$\le F_1 \le \left[\frac{1}{2y}+O\left(\frac{1}{g(y)}\right)\right]\exp\left(\frac{-a_1K^2y}{2f(\pi)\arctan\left(\frac{g(y)}{y}\right)}\right).$$

Now, for $i=1,2$ we have

$$\left[\frac{1}{2y}+O\left(\frac{1}{g(y)}\right)\right]\exp\left(\frac{-a_iK^2y}{2f(\pi)\arctan\left(\frac{g(y)}{y}\right)}\right)$$
$$=\frac{1}{2y}\exp\left(\frac{-a_iK^2y}{2f(\pi)\arctan\left(\frac{g(y)}{y}\right)}\right)+O\left(\frac{1}{g(y)}\right).$$

Thus,

$$\frac{1}{\pi}\int_{\frac{\log\log n}{n}}^{\frac{1}{\log n}}\left[\frac{1}{2y}\exp\left(\frac{-a_iK^2y}{2f(\pi)\arctan\left(\frac{g(y)}{y}\right)}\right)+O\left(\frac{1}{g(y)}\right)\right]dy$$
$$=\frac{1}{\pi}\int_{\frac{\log\log n}{n}}^{\frac{1}{\log n}}\frac{1}{2y}\exp\left(-cK^2y\right)dy+O\left(\log\log n\right)$$
$$\left(\text{where } c=a_i\left[2f(\pi)\arctan\left(\frac{g(y)}{y}\right)\right]^{-1}\right)$$
$$=\frac{1}{2\pi}\left[\log\left(cK^2\frac{1}{\log n}\right)-\log\left(cK^2\frac{\log\log n}{n}\right)\right]$$
$$+\frac{1}{2\pi}\int_0^{cK^2\frac{\log\log n}{n}}\frac{1-e^{-t}}{t}dt-\frac{1}{2\pi}\int_0^{cK^2\frac{1}{\log n}}\frac{1-e^{-t}}{t}dt$$
$$=\frac{1}{2\pi}\log n+\frac{1}{2\pi}\int_0^{cK^2\frac{\log\log n}{n}}\frac{1-e^{-t}}{t}dt-\frac{1}{2\pi}\int_0^{cK^2\frac{1}{\log n}}\frac{1-e^{-t}}{t}dt+O\left(\log\log n\right).$$

Since we are assuming that $K^2\frac{\log\log n}{n}\to 0$ as $n\to\infty$, the first integral is $o(1)$. For the second,

$$=-\frac{1}{2\pi}\int_1^{cK^2\frac{1}{\log n}}\frac{1-e^{-t}}{t}dt-\frac{1}{2\pi}\int_0^1\frac{1-e^{-t}}{t}dt$$
$$=-\frac{1}{2\pi}\int_1^{cK^2\frac{1}{\log n}}\frac{1}{t}dt+\frac{1}{2\pi}\int_1^{cK^2\frac{1}{\log n}}\frac{e^{-t}}{t}dt+O(1)$$
$$=-\frac{1}{2\pi}\log K^2+O\left(\log\log n\right).$$

It follows that

$$\frac{1}{\pi}\int_{-1+\frac{\log\log n}{n}}^{-1+\frac{1}{\log n}} F_1 = \frac{1}{2\pi}\log\left(\frac{n}{K^2}\right) + O\left(\log\log n\right). \tag{3.17}$$

3.3 Expected Number of Level Crossings on $(-\infty,-1)$ and $(1,\infty)$

Now that we have derived the expected number of zeros for $(-1,1)$, in this last section we will consider the remaining intervals $(-\infty,-1)$ and $(1,\infty)$. We will start with the latter. Let $x=\frac{1}{z}$. Then, for $z\in(0,1)$ we have

$$\begin{aligned}
A\left(\frac{1}{z}\right) &= \sum_{k=0}^{n}\sum_{j=0}^{n}\Gamma(k-j)z^{-(k+j)} \\
&= \int_{-\pi}^{\pi}\frac{1-z^{-(n+1)}e^{-i(n+1)\phi}}{1-z^{-1}e^{-i\phi}}\cdot\frac{1-z^{-(n+1)}e^{i(n+1)\phi}}{1-z^{-1}e^{i\phi}}f(\phi)d\phi \\
&= z^{-2n}\int_{-\pi}^{\pi}\frac{1-z^{n+1}e^{i(n+1)\phi}}{1-ze^{i\phi}}\cdot\frac{1-z^{n+1}e^{-i(n+1)\phi}}{1-ze^{-i\phi}}f(\phi)d\phi,
\end{aligned} \tag{3.18}$$

(3.19)

$$\begin{aligned}
B\left(\frac{1}{z}\right) &= \sum_{k=0}^{n}\sum_{j=0}^{n}\Gamma(k-j)kz^{-(k+j-1)} \\
&= \int_{-\pi}^{\pi}\frac{1-z^{-(n+1)}e^{-i(n+1)\phi}}{1-z^{-1}e^{-i\phi}} \\
&\quad\cdot\frac{-(n+1)z^{-n}e^{i(n+1)\phi}\left(1-z^{-1}e^{i\phi}\right)+\left(1-z^{-(n+1)}e^{i(n+1)\phi}\right)e^{i\phi}}{\left(1-z^{-1}e^{i\phi}\right)^2}f(\phi)d\phi \\
&= -z^{-2n+1}\int_{-\pi}^{\pi}\frac{1-z^{n+1}e^{i(n+1)\phi}}{1-ze^{i\phi}} \\
&\quad\cdot\frac{-(n+1)\left(1-ze^{-i\phi}\right)+1-z^{n+1}e^{-i(n+1)\phi}}{\left(1-ze^{-i\phi}\right)^2}f(\phi)d\phi,
\end{aligned}$$

and

(3.20)

$$\begin{aligned}
C\left(\frac{1}{z}\right) &= \sum_{k=0}^{n}\sum_{j=0}^{n}\Gamma(k-j)kjz^{-(k+j-2)} \\
&= \int_{-\pi}^{\pi}\frac{-(n+1)z^{-n}e^{-i(n+1)\phi}\left(1-z^{-1}e^{-i\phi}\right)+\left(1-z^{-(n+1)}e^{-i(n+1)\phi}\right)e^{-i\phi}}{\left(1-z^{-1}e^{-i\phi}\right)^2}
\end{aligned}$$

$$\cdot \frac{-(n+1)z^{-n}e^{i(n+1)\phi}\left(1-z^{-1}e^{i\phi}\right)+\left(1-z^{-(n+1)}e^{i(n+1)\phi}\right)e^{i\phi}}{\left(1-z^{-1}e^{i\phi}\right)^2}f(\phi)d\phi$$

$$= z^{-2n+2}\int_{-\pi}^{\pi}\frac{-(n+1)\left(1-ze^{i\phi}\right)+1-z^{n+1}e^{i(n+1)\phi}}{\left(1-ze^{i\phi}\right)^2}$$

$$\cdot \frac{-(n+1)\left(1-ze^{-i\phi}\right)+1-z^{n+1}e^{-i(n+1)\phi}}{\left(1-ze^{-i\phi}\right)^2}f(\phi)d\phi.$$

As before, our first step is to get a bound for the integral of F_2.

Lemma 3.3.1.

$$\int_1^\infty F_2 dx = \int_{-\infty}^{-1} F_2 dx = o(1).$$

Proof. We have

$$\begin{aligned}\int_1^\infty F_2 dx &\le \frac{\sqrt{2}}{\pi}\int_1^\infty \frac{|B(x)K|}{A^{3/2}(x)}dx \\ &= \frac{\sqrt{2}}{\pi}\int_0^1 \frac{1}{z^2}\frac{\left|B\left(\frac{1}{z}\right)K\right|}{A^{3/2}\left(\frac{1}{z}\right)}dz.\end{aligned} \tag{3.21}$$

Let c_1 and c_2 be as in the proof of Lemma 3.2.1. Then, for $z \in (-1,0)\cup(0,1)$

$$\begin{aligned}\left|B\left(\frac{1}{z}\right)\right| &\le n|z|^{-2n+1}\sum_{k=0}^{n}\sum_{j=0}^{n}\Gamma(k-j)|z|^{2n-k-j} \\ &= n|z|^{-2n+1}A(|z|) \\ &\le c_1 n|z|^{-2n+1}\frac{1-z^{2n+2}}{1-z^2},\end{aligned}$$

where the last line is given by (3.3). Also,

$$\begin{aligned}A\left(\frac{1}{z}\right) &\ge z^{-2n}\frac{c_2}{2\pi}\int_{-\pi}^{\pi}\frac{1-z^{n+1}e^{i(n+1)\phi}}{(1-ze^{i\phi})}\cdot\frac{1-z^{n+1}e^{-i(n+1)\phi}}{(1-ze^{-i\phi})}d\phi \\ &= c_2 z^{-2n}\frac{1-z^{2n+2}}{1-z^2}.\end{aligned}$$

Thus,

$$\frac{\left|B\left(\frac{1}{z}\right)\right|}{A^{3/2}\left(\frac{1}{z}\right)} \le cn|z|^{n+1}\sqrt{\frac{1-z^2}{1-z^{2n+2}}}.$$

Consider the interval $(1-\frac{1}{\sqrt{n}},1)$. Recalling that $K = o\left(\sqrt{\frac{n}{\log\log n}}\right)$, the above inequality yields

$$\frac{\sqrt{2}}{\pi}\int_0^{1-\frac{1}{\sqrt{n}}}\frac{1}{z^2}\frac{\left|B\left(\frac{1}{z}\right)K\right|}{A^{3/2}\left(\frac{1}{z}\right)}dz$$

$$\leq c|K| \int_0^{1-\frac{1}{\sqrt{n}}} nz^{n-1}\sqrt{\frac{1-z^2}{1-z^{2n+2}}}$$
$$\leq c|K| \left(1-\frac{1}{\sqrt{n}}\right)^n$$
$$= o(1).$$

Next, for $z \in (1-\frac{1}{\sqrt{n}}, 1)$ we have

$$\frac{\sqrt{2}}{\pi}\int_{1-\frac{1}{\sqrt{n}}}^1 \frac{1}{z^2}\frac{\left|B\left(\frac{1}{z}\right)K\right|}{A^{3/2}\left(\frac{1}{z}\right)}dz$$
$$\leq c|K| \int_{1-\frac{1}{\sqrt{n}}}^1 nz^{n-1}\sqrt{\frac{1-z^2}{1-z^{2n+2}}}$$
$$= c|K|z^n\sqrt{\frac{1-z^2}{1-z^{2n+2}}}\Bigg|_{1-\frac{1}{\sqrt{n}}}^1 - c|K|\int_{1-\frac{1}{\sqrt{n}}}^1 z^n \frac{d}{dz}\left(\sqrt{\frac{1-z^2}{1-z^{2n+2}}}\right)dz$$
$$= o(1),$$

where the last line follows from the fact that

$$\frac{d}{dz}\left(\sqrt{\frac{1-z^2}{1-z^{2n+2}}}\right) = O\left(\sqrt{n}\right)$$

on $z \in (1-\frac{1}{\sqrt{n}}, 1)$. Applying (3.21), this proves the result for $(1,\infty)$. Noting that the same argument works for $-z$, the result then follows for $(-\infty,-1)$ as well. □

Our next lemma will evaluate the integral of F_1.

Lemma 3.3.2. *(i) For $f \in C([-\pi,\pi])$,*

$$\int_1^\infty F_1 dx = \int_{-\infty}^{-1} F_1 dx \sim \frac{1}{2\pi}\log n.$$

(ii) For $f \in C^1([-\pi,\pi])$,

$$\int_1^\infty F_1 dx = \int_{-\infty}^{-1} F_1 dx = \frac{1}{2\pi}\log n + O(\log\log n).$$

Proof. We will prove the result assuming that $f \in C^1([-\pi,\pi])$; the resulting argument will require only a few minor changes to prove the claim for $f \in C([-\pi,\pi])$. To start, we have the inequality

$$\int_1^\infty F_1 dx \leq \frac{1}{\pi}\int_1^\infty \frac{\sqrt{A(x)C(x)-B^2(x)}}{A(x)}dx.$$

Notice that the expression on the right is simply the expected number of real zeros of $P_n(x)$ on $(1,\infty)$. Similarly,

$$\int_{-\infty}^{-1} F_1 dx \le \frac{1}{\pi}\int_{-\infty}^{-1} \frac{\sqrt{A(x)C(x)-B^2(x)}}{A(x)}dx.$$

Thus, our work in Chapter 2 yields the upper bounds

$$\begin{aligned}\int_1^{\infty} F_1 dx &\le \frac{1}{2\pi}\log n + O\left(\log\log n\right),\\ \int_{-\infty}^{-1} F_1 dx &\le \frac{1}{2\pi}\log n + O\left(\log\log n\right).\end{aligned} \tag{3.22}$$

The rest of the proof will be devoted to the derivation of a lower bound.

Consider the interval $(1-\frac{1}{\log n}, 1-\frac{\log\log n}{n})$. Let $z = 1-y$, and recall that $g(y) = y\frac{\log n}{\log\log n}$. Using (3.18), (3.19), (3.20), and our results from Chapter 2, along with some tedious algebra, we can derive the expression

$$\begin{aligned}
&A\left(\frac{1}{z}\right)C\left(\frac{1}{z}\right) - B^2\left(\frac{1}{z}\right)\\
&= z^{-4n+2}\Bigg[\int_{-\pi}^{\pi}\frac{f(\phi)d\phi}{(1-ze^{i\phi})(1-ze^{-i\phi})}\cdot\int_{-\pi}^{\pi}\frac{f(\phi)d\phi}{(1-ze^{i\phi})^2(1-ze^{-i\phi})^2}\\
&\quad - \left(\int_{-\pi}^{\pi}\frac{f(\phi)d\phi}{(1-ze^{i\phi})(1-ze^{-i\phi})^2}\right)^2\\
&\quad + O\left((n+1)z^{n+1}\int_{-\pi}^{\pi}\frac{f(\phi)d\phi}{(1-ze^{i\phi})(1-ze^{-i\phi})}\cdot\int_{-\pi}^{\pi}\frac{f(\phi)d\phi}{(1-ze^{i\phi})(1-ze^{-i\phi})^2}\right)\Bigg]\\
&= (1-y)^{-4n+2}\left[\frac{f^2(0)}{y^4}\arctan^2\left(\frac{g(y)}{y}\right) + O\left(\frac{1}{y^3 g(y)}\right)\right].
\end{aligned}$$

Thus,

$$\frac{\sqrt{A\left(\frac{1}{z}\right)C\left(\frac{1}{z}\right)-B^2\left(\frac{1}{z}\right)}}{A\left(\frac{1}{z}\right)} = (1-y)\left[\frac{1}{2y} + O\left(\frac{1}{g(y)}\right)\right]. \tag{3.23}$$

Also, if we refer to our work in Chapter 2 once more,

$$C\left(\frac{1}{z}\right) \sim (1-y)^{-2n+2}\frac{2(n+1)^2 f(0)}{y}\arctan\left(\frac{g(y)}{y}\right). \tag{3.24}$$

Applying (3.1) we then have

$$\int_1^{\infty} F_1 dx =$$

$$
\begin{aligned}
&= \frac{1}{\pi}\int_0^1 \frac{1}{z^2}\frac{\sqrt{A\left(\frac{1}{z}\right)C\left(\frac{1}{z}\right)-B^2\left(\frac{1}{z}\right)}}{A\left(\frac{1}{z}\right)}\exp\left(-\frac{K^2C\left(\frac{1}{z}\right)}{2\left(A\left(\frac{1}{z}\right)C\left(\frac{1}{z}\right)-B^2\left(\frac{1}{z}\right)\right)}\right)dz \\
&\geq \frac{1}{\pi}\int_{1-\frac{1}{\log n}}^{1-\frac{\log\log n}{n}} \frac{1}{z^2}\frac{\sqrt{A(\frac{1}{z})C(\frac{1}{z})-B^2(\frac{1}{z})}}{A(\frac{1}{z})}\exp\left(-\frac{K^2C(\frac{1}{z})}{2\left(A(\frac{1}{z})C(\frac{1}{z})-B^2(\frac{1}{z})\right)}\right)dz \\
&= \frac{1}{\pi}\int_{\frac{\log\log n}{n}}^{\frac{1}{\log n}}\left[\frac{1}{2y(1-y)}\left[1+O\left(K^2(n+1)^2(1-y)^{2n}y^3\right)\right]+O\left(\frac{1}{g(y)}\right)\right]dy \\
&= \frac{1}{\pi}\int_{\frac{\log\log n}{n}}^{\frac{1}{\log n}}\frac{1}{2y(1-y)}dy+O\left(\log\log n\right) \\
&= \frac{1}{2\pi}\log n+O\left(\log\log n\right).
\end{aligned}
$$

Noting that almost the exact same argument holds for $-z$,

$$
\int_{-\infty}^{-1} F_1 dx \geq \frac{1}{2\pi}\log n + O\left(\log\log n\right),
$$

as well. Combined with (3.22), the claim then follows. □

Proof of Theorem 3.1.1. Combining the results of Lemmas 3.2.1, 3.2.2, 3.3.1, and 3.3.2, along with equations (3.11) and (3.17), Theorem 3.1.1 now follows. □

3.4 Conclusions

Under the restrictions imposed on the spectral density, we have shown that for the K-level crossings, just as in the case of the expected number of real zeros, behavior similar to the independent case holds. However, similar open questions also remain. That is, it would be of interest to see what happens to the expected number of K-level crossings for covariance functions with slower rates of decay, as well as to identify at what point the behavior changes to match that of the constant covariance case.

Chapter 4

Complex Zeros

4.1 Real Gaussian Coefficients

In this chapter we will extend the work of Shepp and Vanderbei [28] to study the complex zeros of random polynomials with dependent coefficients. This work will be divided into two separate cases. The first will assume dependent standard normal coefficients. Under certain restrictions on the spectral density, we will show that in the limit the zeros accumulate around the unit circle in the complex plane, uniformly in the angle, just as in the independent case. While this result is covered in [15], we present it here for two reasons. First, the analysis here will give a slightly more detailed picture of the way in which this happens. Secondly, we will then employ similar techniques in the second half of this chapter to study a problem which has applications to the GSM (Global System for Mobile Communications)/EDGE (Enhanced Data Rates for GSM Evolution) standard for mobile phones. Here, we will consider random polynomials with dependent complex Gaussian coefficients, having mean zero and exponentially increasing or decreasing variances. As a further note, to give an illustration of this type of behavior in action, in Appendix B we have included a few results of numerical simulations for the independent case of standard normal coefficients.

4.1.1 Formula for the Distribution of Zeros

Before we state our first result we must mention a couple of things. The first concerns the distribution of the zeros of $P_n(z)$. If we consider the function

$$\begin{aligned} z^n P_n(\frac{1}{z}) &= z^n(X_0 + X_1\frac{1}{z} + \cdots + X_n\frac{1}{z^n}) \\ &= X_0 z^n + X_1 z^{n-1} + \cdots + X_n, \end{aligned}$$

it can be seen that whenever $|z_0| > 1$ is a zero of $z^n P_n(\frac{1}{z})$, then $\tilde{z}_0 = \frac{1}{z_0}$ is a zero of $P_n(z)$, where $|\tilde{z}_0| < 1$. Since the distributions of the zeros of the two functions are the same, it is sufficient to only look at the expected number of zeros on the unit disk, and then multiply that amount by two.

We will also need to compute several expressions. Making use of the spectral density form of the covariance function (1.2), we can derive formulas for the following covariances:

$$\begin{aligned} A_0(z) &= \mathrm{E}[P_n^2(z)] = \sum_{k=0}^{n}\sum_{j=0}^{n}\int_{-\pi}^{\pi} f(\phi)e^{-i(k-j)\phi}z^{k+j}d\phi \\ &= \int_{-\pi}^{\pi} f(\phi)\frac{1-z^{n+1}e^{-i(n+1)\phi}}{1-ze^{-i\phi}}\cdot\frac{1-z^{n+1}e^{i(n+1)\phi}}{1-ze^{i\phi}}d\phi, \\ B_0(z) &= \mathrm{E}[P_n(z)P_n(\overline{z})] = \sum_{k=0}^{n}\sum_{j=0}^{n}\int_{-\pi}^{\pi} f(\phi)e^{-i(k-j)\phi}z^k\overline{z}^j d\phi \\ &= \int_{-\pi}^{\pi} f(\phi)\frac{1-z^{n+1}e^{-i(n+1)\phi}}{1-ze^{-i\phi}}\cdot\frac{1-\overline{z}^{n+1}e^{i(n+1)\phi}}{1-\overline{z}e^{i\phi}}d\phi, \\ A_1(z) &= \mathrm{E}[P_n(z)zP_n'(z)] = \sum_{k=0}^{n}\sum_{j=0}^{n}\int_{-\pi}^{\pi} f(\phi)e^{-i(k-j)\phi}jz^{k+j}d\phi \\ &= \int_{-\pi}^{\pi} f(\phi)\frac{1-z^{n+1}e^{-i(n+1)\phi}}{1-ze^{-i\phi}} \\ &\quad\cdot\frac{-(n+1)z^{n+1}e^{i(n+1)\phi}(1-ze^{i\phi})+ze^{i\phi}(1-z^{n+1}e^{i(n+1)\phi})}{(1-ze^{i\phi})^2}d\phi, \\ B_1(z) &= \mathrm{E}[P_n(z)\overline{z}P_n'(\overline{z})] = \sum_{k=0}^{n}\sum_{j=0}^{n}\int_{-\pi}^{\pi} f(\phi)e^{-i(k-j)\phi}jz^k\overline{z}^j d\phi \\ &= \int_{-\pi}^{\pi} f(\phi)\frac{1-z^{n+1}e^{-i(n+1)\phi}}{1-ze^{-i\phi}} \\ &\quad\cdot\frac{-(n+1)\overline{z}^{n+1}e^{i(n+1)\phi}(1-\overline{z}e^{i\phi})+\overline{z}e^{i\phi}(1-\overline{z}^{n+1}e^{i(n+1)\phi})}{(1-\overline{z}e^{i\phi})^2}d\phi. \end{aligned} \tag{4.1}$$

In addition, let

$$D_0(z) = \sqrt{B_0^2(z) - |A_0(z)|^2}. \tag{4.2}$$

One main difference from the independent case is that these expressions are not straightforward to compute; they depend on the values of the spectral density. To apply these formulas we will rely heavily on deriving asymptotic values throughout this section. Let $\nu_n(\Omega)$ be the number of zeros in the set Ω. Our first theorem is stated as follows:

Theorem 4.1.1. *For any region $\Omega \in \mathbb{C}$ whose boundary intersects the real axis at most finitely many times we have*

$$\mathrm{E}[\nu_n(\Omega)] = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{1}{z} F(z) dz, \tag{4.3}$$

where

$$F = \frac{\overline{B}_1 D_0 + B_0 \overline{B}_1 - \overline{A}_0 A_1}{D_0(B_0 + D_0)}. \tag{4.4}$$

Proof. The proof will be based on that of Shepp and Vanderbei, with the addition of some necessary changes to adapt it to the case when the coefficients are no longer assumed to be independent. We will first outline the beginning part of their procedure; the second half of our argument will discuss the needed changes. To start, we can use the argument principle to compute $\nu_n(\Omega)$ (see [6] for a reference). It follows that

$$\nu_n(\Omega) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{P_n'(z)}{P_n(z)} dz.$$

By applying Fubini's Theorem [22] and a result of Hammersley [14] on the distribution of the zeros of a random polynomial with Gaussian coefficients, we can take the expectation and move it inside the integral. Thus, this becomes

$$\begin{aligned}\mathrm{E}[\nu_n(\Omega)] &= \frac{1}{2\pi i}\int_{\partial\Omega} \mathrm{E}\left[\frac{P_n'(z)}{P_n(z)}\right] dz \\ &= \frac{1}{2\pi i}\int_{\partial\Omega} \frac{1}{z}\mathrm{E}\left[\frac{zP_n'(z)}{P_n(z)}\right] dz.\end{aligned}$$

The rest of the proof will be devoted to deriving the formula for $F(z) = \mathrm{E}\left[\frac{zP_n'(z)}{P_n(z)}\right]$ given in the statement of the theorem.

Following the established procedure, we can decompose $P_n(z)$ and $P_n'(z)$ into their real and imaginary parts. We have

$$P_n(z) = Y_1 + iY_2,$$
$$zP_n'(z) = Y_3 + iY_4,$$

where

$$\begin{array}{rclrcl} Y_1 &=& \sum_{j=0}^n a_j X_j, & Y_2 &=& \sum_{j=0}^n b_j X_j, \\ Y_3 &=& \sum_{j=0}^n c_j X_j, & Y_4 &=& \sum_{j=0}^n d_j X_j. \end{array}$$

Furthermore,

$$\begin{array}{rclcl} a_j &=& \mathrm{Re}(z^j) &=& \frac{z^j + \bar{z}^j}{2}, \\ b_j &=& \mathrm{Im}(z^j) &=& \frac{z^j - \bar{z}^j}{2}, \\ c_j &=& ja_j, && \\ d_j &=& jb_j. && \end{array} \tag{4.5}$$

Let M denote the covariance matrix of $Y = \begin{bmatrix} Y_1 & Y_2 & Y_3 & Y_4 \end{bmatrix}^T$. Using the Cholesky factor, L, for the matrix M, we have the decomposition

$$\mathrm{E}\left[YY^T\right] = M = LL^T = \mathrm{E}\left[LWW^TL^T\right],$$

where $W = \begin{bmatrix} W_1 & W_2 & W_3 & W_4 \end{bmatrix}^T$ is a vector of four independent standard normal random variables. In addition to L being lower triangular, notice that the above series of equalities also implies that $Y =_d LW$. That is, Y and LW are equal in distribution. Using these results, we have

$$\begin{aligned} \frac{zP_n'(z)}{P_n(z)} &= \frac{Y_3 + iY_4}{Y_2 + iY_2} \\ &=_d \frac{(l_{31} + il_{41})W_1 + (l_{32} + il_{42})W_2 + (l_{33} + il_{43})W_3 + il_{44}W_4}{(l_{11} + il_{21})W_1 + il_{22}W_2}, \end{aligned}$$

where the l_{ij} are elements of L. From here, after a series of manipulations and calculations (the details of which are in [28]), we arrive at the following formula:

$$\begin{aligned} F(z) &= \mathrm{E}\left[\frac{zP_n'(z)}{P_n(z)}\right] \\ &= \frac{l_{32} - l_{41} + i(l_{31} + l_{42})}{-l_{21} + i(l_{11} + l_{22})}. \end{aligned} \tag{4.6}$$

Up until this point we have basically given a summary of the techniques used by Shepp and Vanderbei. However, we will now need to derive expressions for the elements of L and M, and this is where our argument will begin to diverge from theirs. The dependence among the coefficients of $P_n(z)$ and $P_n'(z)$ presents a slightly different challenge and necessitates the use of the spectral density of the covariance function. The relevant elements of L are

$$\begin{aligned} l_{11} &= \tfrac{m_{11}}{\sqrt{m_{11}}}, \\ l_{21} &= \tfrac{m_{21}}{\sqrt{m_{11}}}, \quad l_{22} = \tfrac{m_{11}m_{22}-m_{21}^2}{R\sqrt{m_{11}}}, \\ l_{31} &= \tfrac{m_{31}}{\sqrt{m_{11}}}, \quad l_{32} = \tfrac{m_{11}m_{32}-m_{31}m_{21}}{R\sqrt{m_{11}}}, \\ l_{41} &= \tfrac{m_{41}}{\sqrt{m_{11}}}, \quad l_{42} = \tfrac{m_{11}m_{42}-m_{41}m_{21}}{R\sqrt{m_{11}}}, \end{aligned}$$

where

$$R = \sqrt{m_{11}m_{22} - m_{21}^2}.$$

Plugging these into (4.6) gives the formula

$$F(z) = \frac{-m_{41} + im_{31} - \frac{i(m_{11}(-m_{42}+im_{32})-m_{21}(-m_{41}+im_{31}))}{R}}{-m_{21} + im_{11} + iR}. \tag{4.7}$$

Our next step is to compute the elements of M. Using the spectral density we can derive the following expressions:

$$\begin{aligned} m_{11} &= \mathrm{E}[Y_1^2] \\ &= \frac{1}{4}\int_{-\pi}^{\pi} f(\phi)\left[\sum_{k=0}^{n}\sum_{j=0}^{n} e^{-ik\phi}e^{ij\phi}(z^{k+j} + \overline{z}^k z^j + z^k\overline{z}^j + \overline{z}^{k+j})\right] d\phi, \\ m_{12} &= \mathrm{E}[Y_1Y_2] \\ &= -\frac{i}{4}\int_{-\pi}^{\pi} f(\phi)\left[\sum_{k=0}^{n}\sum_{j=0}^{n} e^{-ik\phi}e^{ij\phi}(z^{k+j} + z^k\overline{z}^j - \overline{z}^k z^j - \overline{z}^{k+j})\right] d\phi, \\ m_{13} &= \mathrm{E}[Y_1Y_3] \\ &= \frac{1}{4}\int_{-\pi}^{\pi} f(\phi)\left[\sum_{k=0}^{n}\sum_{j=0}^{n} e^{-ik\phi}e^{ij\phi}j(z^{k+j} + \overline{z}^k z^j + z^k\overline{z}^j + \overline{z}^{k+j})\right] d\phi, \end{aligned}$$

$$m_{14} = \mathrm{E}[Y_1Y_4]$$

$$= -\frac{i}{4}\int_{-\pi}^{\pi} f(\phi)\left[\sum_{k=0}^{n}\sum_{j=0}^{n} e^{-ik\phi}e^{ij\phi}j(z^{k+j}+\overline{z}^k z^j - z^k\overline{z}^j - \overline{z}^{k+j})\right] d\phi,$$

$$m_{22} = \mathrm{E}[Y_2^2]$$

$$= -\frac{1}{4}\int_{-\pi}^{\pi} f(\phi)\left[\sum_{k=0}^{n}\sum_{j=0}^{n} e^{-ik\phi}e^{ij\phi}(z^{k+j}-\overline{z}^k z^j - z^k\overline{z}^j + \overline{z}^{k+j})\right] d\phi,$$

$$m_{23} = \mathrm{E}[Y_2Y_3]$$

$$= -\frac{i}{4}\int_{-\pi}^{\pi} f(\phi)\left[\sum_{k=0}^{n}\sum_{j=0}^{n} e^{-ik\phi}e^{ij\phi}j(z^{k+j}+z^k\overline{z}^j - \overline{z}^k z^j - \overline{z}^{k+j})\right] d\phi,$$

$$m_{24} = \mathrm{E}[Y_2Y_4]$$

$$= -\frac{1}{4}\int_{-\pi}^{\pi} f(\phi)\left[\sum_{k=0}^{n}\sum_{j=0}^{n} e^{-ik\phi}e^{ij\phi}j(z^{k+j}-z^k\overline{z}^j - \overline{z}^k z^j + \overline{z}^{k+j})\right] d\phi.$$

Using (4.1), with a little work we can now express the elements of M as follows:

$$\begin{array}{lllll}
m_{11} & = & \mathrm{E}[\xi_1^2] & = & \frac{1}{4}(A_0+2B_0+\overline{A}_0) \\
m_{12} & = & \mathrm{E}[\xi_1\xi_2] & = & -\frac{i}{4}(A_0-\overline{A}_0) \\
m_{13} & = & \mathrm{E}[\xi_1\xi_3] & = & \frac{1}{4}(A_1+B_1+\overline{B}_1+\overline{A}_1) \\
m_{14} & = & \mathrm{E}[\xi_1\xi_4] & = & -\frac{i}{4}(A_1+\overline{B}_1-B_1-\overline{A}_1) \\
m_{22} & = & \mathrm{E}[\xi_2^2] & = & -\frac{1}{4}(A_0-2B_0+\overline{A}_0) \\
m_{23} & = & \mathrm{E}[\xi_2\xi_3] & = & -\frac{i}{4}(A_1+B_1-\overline{B}_1-\overline{A}_1) \\
m_{24} & = & \mathrm{E}[\xi_2\xi_4] & = & -\frac{1}{4}(A_1-B_1-\overline{B}_1+\overline{A}_1).
\end{array} \tag{4.8}$$

Applying (4.2) and a little algebra, we also have

$$\begin{aligned}
R &= \sqrt{m_{11}m_{22}-m_{21}^2} \\
&= \frac{1}{2}\sqrt{B_0^2-|A_0|^2} \\
&= \frac{1}{2}D_0.
\end{aligned} \tag{4.9}$$

Finally, by combining (4.7), (4.8), and (4.9), along with some tedious simplifications, we arrive at the formula

$$\begin{aligned}
F(z) &= \frac{(A_0+2B_0+\overline{A}_0)(\overline{B}_1D_0+B_0\overline{B}_1-\overline{A}_0A_1)}{(A_0+2B_0+\overline{A}_0)D_0(B_0+D_0)} \\
&= \frac{\overline{B}_1D_0+B_0\overline{B}_1-\overline{A}_0A_1}{D_0(B_0+D_0)}
\end{aligned}$$

$$= \frac{\overline{B}_1 D_0 + B_0 \overline{B}_1 - \overline{A}_0 A_1}{B_0 D_0 + B_0^2 - \overline{A}_0 A_0},$$

as claimed. □

4.1.2 Properties of the Distribution of Zeros

Now that we have verified Shepp and Vanderbei's formula for the expected number of zeros when some dependence is assumed among the coefficients, we can discuss some applications. We will proceed as they did, proving a series of results which illustrates the interesting behavior of the zeros. While we are expecting similar behavior as in the independent case, the extra assumption of dependence will force us to rely on the spectral density form of the covariance function, along with several asymptotic results, to show this. We will start by proving two theorems which discuss the accumulation of zeros around the unit circle.

Theorem 4.1.2. *Let $D(r)$ be the disk of radius r centered at 0. For any $s \geq 0$ we have*

$$\lim_{n\to\infty} \mathrm{E}\left[\frac{1}{n+1}\nu_n\left(D\left(e^{-s/2(n+1)}\right)\right)\right] = \frac{1-e^{-s}(1+s)}{s(1-e^{-s})}$$
$$\sim \frac{1}{2} - \frac{s}{3}, \quad s \to 0.$$

Proof. From (4.3) we have

$$\mathrm{E}[\nu_n\left(D(r)\right)] = \frac{1}{2\pi i}\int_{\partial D(r)} \frac{1}{z}F(z)dz$$
$$= \frac{1}{2\pi}\int_0^{2\pi} F(re^{i\theta})d\theta,$$

where

$$r = e^{-s/2(n+1)}, \quad s \geq 0, \quad z = re^{i\theta}.$$

Rewriting (4.4) we can express $F(z)$ as

(4.10) $$F(z) = \frac{\overline{B}_1 + \frac{B_0\overline{B}_1 - \overline{A}_0 A_1}{D_0}}{B_0 + D_0}.$$

We will start by determining the asymptotic behavior of $\overline{B}_1$ and B_0. By then showing that the contribution from the $\overline{A}_0 A_1$ term vanishes in the limit, we will have the asymptotic behavior for $F(z)$ as a whole. Note that we can assume θ is bounded some small distance away from $-\pi$ and π. Otherwise, using the fact that

$$\Gamma(k) = \int_{-\pi}^{\pi} e^{-ik\phi} f(\phi) d\phi = \int_{0}^{2\pi} e^{-ik\phi} f(\phi) d\phi = \int_{-2\pi}^{0} e^{-ik\phi} f(\phi) d\phi$$

for any k, the following results will hold with only minor changes to the arguments used.

For B_0 we have

$$\begin{aligned} B_0(z) &= \int_{-\pi}^{\pi} f(\phi) \frac{1-e^{-s/2}e^{i(n+1)(\theta-\phi)}}{1-e^{-s/2(n+1)}e^{i(\theta-\phi)}} \cdot \frac{1-e^{-s/2}e^{i(n+1)(\phi-\theta)}}{1-e^{-s/2(n+1)}e^{i(\phi-\theta)}} d\phi \\ &= \int_{\theta-(n+1)^{-\frac{1}{4}}}^{\theta+(n+1)^{-\frac{1}{4}}} f(\phi) \frac{1-2e^{-s/2}\cos\left[(n+1)(\theta-\phi)\right]+e^{-s}}{1-2e^{-s/2(n+1)}\cos\left(\theta-\phi\right)+e^{-s/(n+1)}} d\phi \\ &\quad + \int_{\theta+(n+1)^{-\frac{1}{4}}}^{\pi} f(\phi) \frac{1-2e^{-s/2}\cos\left[(n+1)(\theta-\phi)\right]+e^{-s}}{1-2e^{-s/2(n+1)}\cos\left(\theta-\phi\right)+e^{-s/(n+1)}} d\phi \\ &\quad + \int_{-\pi}^{\theta-(n+1)^{-\frac{1}{4}}} f(\phi) \frac{1-2e^{-s/2}\cos\left[(n+1)(\theta-\phi)\right]+e^{-s}}{1-2e^{-s/2(n+1)}\cos\left(\theta-\phi\right)+e^{-s/(n+1)}} d\phi \\ &= B_0^1 + B_0^2 + B_0^3. \end{aligned}$$

Starting with B_0^1,

$$\begin{aligned} B_0^1 &\sim \int_{\theta}^{\theta+(n+1)^{-\frac{1}{4}}} \frac{2c_n f(\phi) d\phi}{2-2(1-\frac{s}{2(n+1)}+\frac{s^2}{8(n+1)^2})(1-\frac{(\theta-\phi)^2}{2})-\frac{s}{n+1}+\frac{s^2}{2(n+1)^2}} \\ &\sim 2c_n f(\theta) \int_{\theta}^{\theta+(n+1)^{-\frac{1}{4}}} \frac{d\phi}{(\theta-\phi)^2+\frac{s^2}{4(n+1)^2}} \\ &= c_n f(\theta) \frac{4}{s}(n+1) \arctan\left(\frac{2}{s}(n+1)(\phi-\theta)\right)\Bigg|_{\theta}^{\theta+(n+1)^{-1/4}} \\ &\sim c_n f(\theta) \frac{2\pi}{s}(n+1). \end{aligned}$$

We will next show that B_0^2 and B_0^3 are small compared to B_0^1. For B_0^2,

$$\begin{aligned} B_0^2 &\sim \int_{\theta+(n+1)^{-\frac{1}{4}}}^{\pi} \frac{cf(\phi)}{1-2e^{-s/2(n+1)}\cos\left(\theta-\phi\right)+e^{-s/(n+1)}} d\phi \\ &\leq \int_{\theta+(n+1)^{-\frac{1}{4}}}^{\pi} \frac{cf(\phi)}{1-2e^{-s/2(n+1)}\cos\left(-(n+1)^{-1/4}\right)+e^{-s/(n+1)}} d\phi \\ &\sim \int_{\theta+(n+1)^{-\frac{1}{4}}}^{\pi} \frac{cf(\phi)}{(n+1)^{-1/2}+\frac{s^2}{4(n+1)^2}} d\phi \end{aligned}$$

$$
\begin{aligned}
&\sim c(n+1)^{1/2}\\
&= o\left(B_0^1\right).
\end{aligned}
$$

Similarly, we can also show that $B_0^3 = o\left(B_0^1\right)$. It follows that

$$B_0(z) \sim B_0^1 \sim c_n f(\theta) \frac{2\pi}{s}(n+1).$$

In the independent case $f(\theta) \equiv \frac{1}{2\pi}$. Setting the quantity above equal to the asymptotic value of $B_0(z)$ in the independent case, $(n+1)\frac{1-e^{-s}}{s}$, allows us to solve for c_n. Thus,

$$(n+1)\frac{c_n}{s} = (n+1)\frac{1-e^{-s}}{s} \Rightarrow c_n = 1-e^{-s},$$

and we can now conclude that

$$\lim_{n\to\infty} \frac{1}{n+1} B_0 = 2\pi \frac{1-e^{-s}}{s} f(\theta).$$

Next, for B_1,

$$
\begin{aligned}
B_1(z) = &\int_{-\pi}^{\pi} \left[\frac{\left(z^{n+1}e^{-i(n+1)\phi} - 1\right)(n+1)\left(\overline{z}e^{i\phi}\right)^{n+1}}{\left(1-ze^{-i\phi}\right)\left(1-\overline{z}e^{i\phi}\right)} \right.\\
&\left. + \frac{\left|1-z^{n+1}e^{-i(n+1)\phi}\right|^2 \left(\overline{z}e^{i\phi} - |z|^2\right)}{\left(1-ze^{-i\phi}\right)^2\left(1-\overline{z}e^{i\phi}\right)^2} \right] f(\phi)d\phi\\
= &\int_{-\pi}^{\pi} f(\phi) \left[\frac{-(n+1)\left(e^{-s/2}e^{i(n+1)(\phi-\theta)} - e^{-s}\right)}{\left(1-e^{-s/2(n+1)}e^{i(\theta-\phi)}\right)\left(1-e^{-s/2(n+1)}e^{i(\phi-\theta)}\right)} \right.\\
&\left. + \frac{\left|1-e^{-s/2}e^{i(n+1)(\theta-\phi)}\right|^2 \left(e^{-s/2(n+1)}e^{i(\phi-\theta)} - e^{-s/(n+1)}\right)}{\left(1-e^{-s/2(n+1)}e^{i(\theta-\phi)}\right)^2\left(1-e^{-s/2(n+1)}e^{i(\phi-\theta)}\right)^2} \right] d\phi\\
\sim &\int_{-\pi}^{\pi} f(\phi) \frac{c_n^1 \cdot (n+1)}{\left(1-e^{-s/2(n+1)}e^{i(\theta-\phi)}\right)\left(1-e^{-s/2(n+1)}e^{i(\phi-\theta)}\right)} d\phi\\
&+ \int_{-\pi}^{\pi} \frac{c_n^2 f(\phi)\left(e^{-s/2(n+1)}e^{i(\phi-\theta)} - e^{-s/(n+1)}\right)}{\left(1-e^{-s/2(n+1)}e^{i(\theta-\phi)}\right)^2\left(1-e^{-s/2(n+1)}e^{i(\phi-\theta)}\right)^2} d\phi\\
= &B_1^1 + B_1^2.
\end{aligned}
$$

From our work on B_0 we know that

$$B_1^1 \sim c_n^1 (n+1)^2 f(\theta) \frac{2\pi}{s}.$$

To handle B_1^2 we can apply a procedure similar to the one used on B_0^1 and B_0^2. We then have

$$
\begin{aligned}
B_1^2 &\sim \int_{\theta-(n+1)^{-\frac{1}{4}}}^{\theta+(n+1)^{-\frac{1}{4}}} f(\phi)\frac{c_n^2\left(e^{-s/2(n+1)}e^{i(\phi-\theta)}-e^{-s/(n+1)}\right)}{\left(1-2e^{-s/2(n+1)}\cos\left(\theta-\phi\right)+e^{-s/(n+1)}\right)^2}d\phi \\
&\sim 2f(\theta)\int_{\theta}^{\theta+(n+1)^{-\frac{1}{4}}} \frac{c_n^2\left(\frac{s}{2(n+1)}-\frac{(\theta-\phi)^2}{2}\right)}{\left((\theta-\phi)^2+\frac{s^2}{4(n+1)^2}\right)^2}d\phi \\
&= 2f(\theta)\frac{c_n^2(n+1)^2}{s^2}\left[\frac{-(4s(n+1)+s^2)(\theta-\phi)}{s^2+4(n+1)^2(\theta-\phi)^2}\right. \\
&\quad \left.\left. + \left(\frac{s}{2(n+1)}-2\right)\arctan\left(\frac{2}{s}(n+1)(\theta-\phi)\right)\right]\right|_{\theta}^{\theta+(n+1)^{-\frac{1}{4}}} \\
&\sim 2\pi f(\theta)(n+1)^2\frac{c_n^2}{s^2}.
\end{aligned}
$$

Since the asymptotic value of $B_1(z)$ in the independent case is $(n+1)^2\frac{1-e^{-s}(1+s)}{s^2}$, we can again solve for the constants using the same procedure as before. Thus,

$$
\frac{c_n^1}{s}+\frac{c_n^2}{s^2}=\frac{1-e^{-s}(1+s)}{s^2},
$$

from which it follows that

$$
\lim_{n\to\infty}\frac{1}{(n+1)^2}B_1 = 2\pi\frac{1-e^{-s}(1+s)}{s^2}f(\theta).
$$

Lastly, since f is real-valued,

$$
\begin{aligned}
\lim_{n\to\infty}\frac{1}{(n+1)^2}\overline{B}_1 &= 2\pi\frac{1-e^{-s}(1+s)}{s^2}\overline{f(\theta)} \\
&= 2\pi\frac{1-e^{-s}(1+s)}{s^2}f(\theta),
\end{aligned}
$$

as well.

Our next step is to show that

$$
\lim_{n\to\infty}\frac{\overline{A}_0A_1}{(n+1)^3}=0. \tag{4.11}
$$

Now, for small $\theta+\phi$ we have

$$
\frac{1}{1-ze^{-i\phi}}=\frac{1-\overline{z}e^{i\phi}}{\left(1-ze^{-i\phi}\right)\left(1-\overline{z}e^{i\phi}\right)}
$$

$$
\begin{aligned}
&= \frac{1 - e^{-s/2(n+1)}\left[\cos(\theta-\phi) - i\sin(\theta-\phi)\right]}{1 + e^{-s/(n+1)} - 2e^{-s/2(n+1)}\cos(\theta-\phi)} \\
&\sim \frac{\frac{(\theta-\phi)^2}{2} + \frac{s}{2(n+1)} + i(\theta-\phi)}{(\theta-\phi)^2 + \frac{s}{2(n+1)}}.
\end{aligned}
$$

Similarly, for $\theta - \phi$ small we have

$$
\begin{aligned}
\frac{1}{1 - ze^{i\phi}} &= \frac{1 - \overline{z}e^{-i\phi}}{\left(1 - ze^{i\phi}\right)\left(1 - \overline{z}e^{-i\phi}\right)} \\
&= \frac{1 - e^{-s/2(n+1)}\left[\cos(\theta+\phi) - i\sin(\theta+\phi)\right]}{1 + e^{-s/(n+1)} - 2e^{-s/2(n+1)}\cos(\theta+\phi)} \\
&\sim \frac{\frac{(\theta+\phi)^2}{2} + \frac{s}{2(n+1)} + i(\theta+\phi)}{(\theta+\phi)^2 + \frac{s}{2(n+1)}}.
\end{aligned}
$$

For A_0 we then have the inequality

$$
\begin{aligned}
A_0 &\sim \int_{-\theta-\frac{1}{\log(n+1)}}^{-\theta+\frac{1}{\log(n+1)}} f(\phi)\frac{c_1}{1 - ze^{-i\phi}}d\phi + \int_{-\theta-\frac{1}{\log(n+1)}}^{-\theta+\frac{1}{\log(n+1)}} f(\phi)\frac{c_2}{1 - ze^{i\phi}}d\phi \\
&\sim c_1\int_{\theta-\frac{1}{\log(n+1)}}^{\theta+\frac{1}{\log(n+1)}} \frac{\frac{(\theta-\phi)^2}{2} + \frac{s}{2(n+1)} + i(\theta-\phi)}{(\theta-\phi)^2 + \frac{s}{2(n+1)}} f(\phi)d\phi \\
&\quad + c_2\int_{\theta-\frac{1}{\log(n+1)}}^{\theta+\frac{1}{\log(n+1)}} \frac{\frac{(\theta+\phi)^2}{2} + \frac{s}{2(n+1)} + i(\theta+\phi)}{(\theta+\phi)^2 + \frac{s}{2(n+1)}} f(\phi)d\phi \\
&\leq c\log(n+1).
\end{aligned}
$$

Similarly, we can also bound A_1 from above:

$$
\begin{aligned}
A_1 &\sim \int_{-\theta-\frac{1}{\log(n+1)}}^{-\theta+\frac{1}{\log(n+1)}} \left[\frac{c_1(n+1)}{1 - ze^{i\phi}} + \frac{c_2}{\left(1 - ze^{i\phi}\right)^2}\right] f(\phi)d\phi \\
&\quad + \int_{\theta-\frac{1}{\log(n+1)}}^{\theta+\frac{1}{\log(n+1)}} \frac{c_3(n+1)}{1 - ze^{-i\phi}} f(\phi)d\phi \\
&\sim \int_{-\theta-\frac{1}{\log(n+1)}}^{-\theta+\frac{1}{\log(n+1)}} \left[c_1(n+1)\frac{\frac{(\theta+\phi)^2}{2} + \frac{s}{2(n+1)} + i(\theta+\phi)}{(\theta+\phi)^2 + \frac{s}{2(n+1)}}\right. \\
&\quad \left. + c_2\left(\frac{\frac{(\theta+\phi)^2}{2} + \frac{s}{2(n+1)} + i(\theta+\phi)}{(\theta+\phi)^2 + \frac{s}{2(n+1)}}\right)^2\right] f(\phi)d\phi \\
&\quad + \int_{\theta-\frac{1}{\log(n+1)}}^{\theta+\frac{1}{\log(n+1)}} c_3(n+1)\frac{\frac{(\theta-\phi)^2}{2} + \frac{s}{2(n+1)} + i(\theta-\phi)}{(\theta-\phi)^2 + \frac{s}{2(n+1)}} f(\phi)d\phi
\end{aligned}
$$

$$\leq c(n+1)\log(n+1).$$

From here it is easy to see that (4.11) holds. Combining these results and applying the Lebesgue dominated convergence theorem (see [21] for a reference) leads to the formula

$$\begin{aligned}
\lim_{n\to\infty} \mathrm{E}\frac{1}{n+1}\nu_n\left(B(r)\right) &= \frac{1}{2\pi}\int_0^{2\pi} \lim_{n\to\infty}\frac{1}{n+1}F(z)d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{2\pi\frac{1-e^{-s}(1+s)}{s^2}f(\theta) + 2\pi\frac{1-e^{-s}(1+s)}{s^2}f(\theta)}{4\pi\frac{1-e^{-s}}{s}f(\theta)}d\phi \\
&= \frac{1-e^{-s}(1+s)}{s(1-e^{-s})} \\
&\sim \frac{1}{2} - \frac{s}{3}, \quad s\to 0,
\end{aligned}$$

as claimed. □

Theorem 4.1.3. *Let* $r = e^{-1/2(k+1)}$. *Then,*

$$\lim_{n\to\infty} \mathrm{E}[\nu_n\left(D(r)\right)] \sim k+1,$$

as $k\to\infty$.

Proof. Applying (4.3) gives us

$$\mathrm{E}[\nu_n(D(r))] = \frac{1}{2\pi}\int_0^{2\pi} F(re^{i\theta})d\theta,$$

from which it follows (along with the Lebesgue dominated convergence theorem) that

$$\lim_{n\to\infty}\mathrm{E}[\nu_n(D(r))] = \frac{1}{2\pi}\int_0^{2\pi}\lim_{n\to\infty}F(re^{i\theta})d\theta.$$

Applying the Lebesgue dominated convergence theorem once more, we then have the formulas

$$\begin{aligned}
A = \lim_{n\to\infty} A_0 &= \int_{-\pi}^{\pi} f(\phi)\frac{1}{1-ze^{-i\phi}}\cdot\frac{1}{1-ze^{i\phi}}d\phi, \\
B = \lim_{n\to\infty} B_0 &= \int_{-\pi}^{\pi} f(\phi)\frac{1}{1-ze^{-i\phi}}\cdot\frac{1}{1-\overline{z}e^{i\phi}}d\phi, \\
A' = \lim_{n\to\infty} A_1 &= \int_{-\pi}^{\pi} f(\phi)\frac{1}{1-ze^{-i\phi}}\cdot\frac{ze^{i\phi}}{(1-ze^{i\phi})^2}d\phi,
\end{aligned}$$

$$B' = \lim_{n\to\infty} B_1 = \int_{-\pi}^{\pi} f(\phi)\frac{1}{1 - ze^{-i\phi}} \cdot \frac{\overline{z}e^{i\phi}}{(1 - \overline{z}e^{i\phi})^2}d\phi,$$

$$D = \lim_{n\to\infty} D_0 = \sqrt{B^2 - |A|^2}.$$

It follows that

$$\lim_{n\to\infty} F(z) = \frac{B'D + BB' - \overline{A}A'}{D(B + D)}.$$

Adding and subtracting $\frac{B'}{B}|A|^2$ from the denominator gives

$$\lim_{n\to\infty} F(z) = \frac{B'}{B} + \frac{|A|^2}{D(B + D)}\left(\frac{B'}{B} - \frac{A'}{A}\right). \tag{4.12}$$

Now, recall that $r = e^{-1/2(k+1)}$. We will next show that the second term stays bounded as k increases. If we repeat the analysis done in the proof of Theorem 4.1.2 for A_0, A_1, B_0 and B_1 on the terms A, A', B and B', respectively, we will obtain similar results. That is,

$$\begin{aligned} B &\sim 4\pi f(\theta)(k + 1), \\ B' &\sim 4\pi f(\theta)(k + 1)^2, \\ A &\leq c_1 \log{(k + 1)}, \\ A' &\leq c_2(k + 1) \log{(k + 1)}, \end{aligned} \tag{4.13}$$

for $c_1, c_2 > 0$, as $k \to \infty$. Applying these results gives the inequality

$$\begin{aligned} \frac{|A|^2}{D(B + D)}\left(\frac{B'}{B} - \frac{A'}{A}\right) &\leq C\frac{(\log{(k + 1)})^2(k + 1)^2}{(k + 1)^3} \\ &\leq C. \end{aligned}$$

Thus, letting k increase we can see that the second term is bounded by a constant. Furthermore, if we plug in these values for B and B' into (4.12), we have

$$\lim_{n\to\infty} F(z) \sim k + 1,$$

as $k \to \infty$. The claim then follows. □

Theorem 4.1.4. *Let $C(\theta_1, \theta_2)$ be the cone in the complex plane consisting of all points with arguments between θ_1 and θ_2. Furthermore, assume $C(\theta_1, \theta_2)$ does not intersect the real axis. Then*

$$\lim_{n\to\infty} \frac{1}{n + 1}\mathrm{E}[\nu_n(C(\theta_1, \theta_2))] = \frac{\theta_1 - \theta_2}{2\pi}.$$

In other words, the zeros are distributed in the complex plane uniformly in the angle.

Proof. It is sufficient to compute the limit for the intersection of $C(\theta_1, \theta_2)$ with the unit disk, and then multiply the result by two. Let $R(\theta_1, \theta_2)$ denote the polar rectangle resulting from this intersection. Applying (4.3) once again we have

$$\begin{aligned} \mathrm{E}[\nu_n(R(\theta_1, \theta_2))] &= \frac{1}{2\pi i} \int_{\partial\Omega} \frac{1}{z} F(z) dz \\ &= \frac{1}{2\pi i} \int_0^1 \frac{1}{r} F(re^{i\theta_1}) dr + \frac{1}{2\pi} \int_{\theta_1}^{\theta_2} F(e^{i\theta}) d\theta \\ &\quad + \frac{1}{2\pi i} \int_1^0 \frac{1}{r} F(re^{i\theta_2}) dr \\ &= \frac{1}{2\pi i} \int_0^1 \frac{1}{r} \left[F(re^{i\theta_1}) - F(re^{i\theta_2}) \right] dr + \frac{1}{2\pi} \int_{\theta_1}^{\theta_2} F(e^{i\theta}) d\theta. \end{aligned} \tag{4.14}$$

Our initial step will be to show that the first integral vanishes in the limit. Let $r > 0$ and rewrite F in the form

$$F(z) = \frac{\overline{B}_1}{D_0} - \frac{\overline{A}_0 A_1}{D_0(B_0 + D_0)}.$$

Then,

$$\begin{aligned} F(re^{i\theta_1}) - F(re^{i\theta_2}) &\sim \frac{\overline{B}_1(re^{i\theta_1})}{D_0(re^{i\theta_1})} - \frac{\overline{B}_1(re^{i\theta_2})}{D_0(re^{i\theta_2})} \\ &= \overline{B}_1(r) \frac{D_0(re^{i\theta_2}) - D_0(re^{i\theta_1})}{D_0(re^{i\theta_1}) D_0(re^{i\theta_2})}, \end{aligned}$$

since $\overline{B}_1$ depends on r only. Analyzing the numerator we have

$$\begin{aligned} &D_0(re^{i\theta_2}) - D_0(re^{i\theta_1}) \\ &\quad = \sqrt{(n+1)^2 - |A_0(re^{i\theta_1})|^2} - \sqrt{(n+1)^2 - |A_0(re^{i\theta_2})|^2} \\ &\quad = (n+1)n \left(\sqrt{1 - \frac{|A_0(re^{i\theta_1})|^2}{n^2}} - \sqrt{1 - \frac{|A_0(re^{i\theta_2})|^2}{n^2}} \right) \\ &\quad \sim n \left(\frac{|A_0(re^{i\theta_1})|^2}{2n^2} - \frac{|A_0(re^{i\theta_2})|^2}{2n^2} \right) \\ &\quad = \frac{1}{2n} \left(|A_0(re^{i\theta_1})|^2 - |A_0(re^{i\theta_2})|^2 \right), \end{aligned} \tag{4.15}$$

from which it then follows that

$$F(re^{i\theta_1}) - F(re^{i\theta_2}) \sim \frac{\overline{B}_1(r)}{D_0(re^{i\theta_1}) D_0(re^{i\theta_2})} \cdot \frac{1}{2n} \left(|A_0(re^{i\theta_1})|^2 - |A_0(re^{i\theta_2})|^2 \right)$$

$$\sim c\left(|A_0(re^{i\theta_1})|^2 - |A_0(re^{i\theta_2})|^2\right).$$

Thus, for $r > 0$, $\frac{1}{n}$ times the integrand vanishes in the limit. Now, to consider the case when $r = 0$ we will first need the following limits:

$$\lim_{r\to 0} B_0 = \lim_{r\to 0} A_0 = 1,$$
$$\lim_{r\to 0} D_0 = 0,$$
$$\lim_{r\to 0} \frac{1}{r}\overline{B}_1 = 1.$$

Letting $r \to 0$ we will also need the expressions

$$\begin{aligned} B_0\overline{B}_1 - \overline{A}_0 A_1 &\sim r^2(1-e^{2i\theta}) - r^2(1-e^{2i\theta})\left(\int_{-\pi}^{\pi} f(\phi)e^{i\phi}d\phi\right)^2 \\ &= r^2(1-e^{2i\theta})\left[1 - \left(\int_{-\pi}^{\pi} f(\phi)e^{i\phi}d\phi\right)^2\right], \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} B_0^2 - |A_0|^2 &\sim 2r^2(1-\cos 2\theta) - 2r^2(1-\cos 2\theta)\left(\int_{-\pi}^{\pi} f(\phi)e^{i\phi}d\phi\right)^2 \\ &= 4r^2\sin^2\theta\left[1 - \left(\int_{-\pi}^{\pi} f(\phi)e^{i\phi}d\phi\right)^2\right]. \end{aligned} \tag{4.17}$$

Combining (4.16) and (4.17), and recalling (4.2), we then have

$$\begin{aligned} \lim_{r\to 0}\frac{1}{r}\frac{B_0\overline{B}_1 - \overline{A}_0 A_1}{D_0} &\sim \lim_{r\to 0}\frac{r(1-e^{2i\theta})\left[1 - \left(\int_{-\pi}^{\pi} f(\phi)e^{i\phi}d\phi\right)^2\right]}{2r\sin\theta\sqrt{1 - \left(\int_{-\pi}^{\pi} f(\phi)e^{i\phi}d\phi\right)^2}} \\ &= \frac{(1-e^{2i\theta})}{\sin\theta}\sqrt{1 - \left(\int_{-\pi}^{\pi} f(\phi)e^{i\phi}d\phi\right)^2}, \end{aligned}$$

which leads to the final expression

$$\begin{aligned} \lim_{r\to 0}\frac{1}{r}F(re^{i\theta}) &= \frac{1}{r}\frac{\overline{B}_1 + \frac{B_0\overline{B}_1 - \overline{A}_0 A_1}{D_0}}{B_0 + D_0} \\ &= 1 + \frac{(1-e^{2i\theta})}{\sin\theta}\sqrt{1 - \left(\int_{-\pi}^{\pi} f(\phi)e^{i\phi}d\phi\right)^2}. \end{aligned}$$

Thus, even at $r = 0$ the integrand is bounded in n. Combining these results, we can conclude that the contribution from the first integral disappears in the limit.

We will now derive the asymptotic value for the second integral in (4.14). For $z = e^{i\theta}$ we have

$$
\begin{aligned}
B_0(e^{i\theta}) &= \int_{-\pi}^{\pi} f(\phi) \frac{1 - e^{i(n+1)(\theta-\phi)}}{1 - e^{i(\theta-\phi)}} \cdot \frac{1 - e^{i(n+1)(\phi-\theta)}}{1 - e^{i(\phi-\theta)}} d\phi \\
&= \int_{-\pi}^{\pi} f(\phi) \frac{1 - e^{i(n+1)(\theta-\phi)} - e^{i(n+1)(\phi-\theta)} + 1}{1 - e^{i(\theta-\phi)} - e^{i(\phi-\theta)} + 1} d\phi \\
&= \int_{-\pi}^{\pi} f(\phi) \frac{1 - \cos\left((n+1)(\theta-\phi)\right)}{1 - \cos\left(\theta-\phi\right)} d\phi \\
&= \int_{\theta-(n+1)^{-1/4}}^{\theta+(n+1)^{-1/4}} f(\phi) \frac{1 - \cos\left((n+1)(\theta-\phi)\right)}{1 - \cos\left(\theta-\phi\right)} d\phi \\
&\quad + \int_{-\pi}^{\theta-(n+1)^{-1/4}} f(\phi) \frac{1 - \cos\left((n+1)(\theta-\phi)\right)}{1 - \cos\left(\theta-\phi\right)} d\phi \\
&\quad + \int_{\theta+(n+1)^{-1/4}}^{\pi} f(\phi) \frac{1 - \cos\left((n+1)(\theta-\phi)\right)}{1 - \cos\left(\theta-\phi\right)} d\phi \\
&= B_0^1 + B_0^2 + B_0^3.
\end{aligned}
$$

For B_0^1,

$$
\begin{aligned}
B_0^1 &= 2\int_{\theta}^{\theta+(n+1)^{-1/4}} f(\phi) \frac{1 - \cos\left((n+1)(\theta-\phi)\right)}{1 - \cos\left(\theta-\phi\right)} d\phi \\
&\sim 4f(\theta) \int_{\theta}^{\theta+(n+1)^{-1/4}} \frac{1 - \cos\left((n+1)(\theta-\phi)\right)}{(\theta-\phi)^2} d\phi \\
&= 4f(\theta) \frac{1}{\phi-\theta} \left[-1 + \cos\left((n+1)(\theta-\phi)\right)\right. \\
&\quad \left. + (n+1)(\theta-\phi)\mathrm{Si}\left((n+1)(\theta-\phi)\right)\right] \Big|_{\theta}^{\theta+(n+1)^{-1/4}} \\
&\sim -4f(\theta)(n+1)\mathrm{Si}\left(-(n+1)^{3/4}\right) \\
&\sim 2\pi f(\theta)(n+1),
\end{aligned}
$$

where $\mathrm{Si}(z) = \int_0^z \frac{\sin t}{t} dt$. For B_0^2 we have

$$
\begin{aligned}
B_0^2 &\leq \int_{-\pi}^{\theta-(n+1)^{-1/4}} \frac{c}{1 - \cos\left((n+1)^{-1/4}\right)} d\phi \\
&\sim \int_{-\pi}^{\theta-(n+1)^{-1/4}} \frac{c}{\frac{(n+1)^{-1/2}}{2}} d\phi \\
&= o\left(n\right).
\end{aligned}
$$

In a similar fashion we can also show that $B_0^3 = o(n)$. Thus, it follows that

$$B_0\left(re^{i\theta}\right) = 2\pi f(\theta)(n+1) + o(n). \tag{4.18}$$

To handle B_1 we will break it up as

$$\begin{aligned}
B_1(e^{i\theta}) &= \int_{-\pi}^{\pi} f(\phi) \frac{-(n+1)e^{i(n+1)(\phi-\theta)} - 1}{\left(1-e^{i(\theta-\phi)}\right)\left(1-e^{i(\phi-\theta)}\right)} d\phi \\
&\quad + \int_{-\pi}^{\pi} f(\phi) \frac{e^{i(\phi-\theta)}\left|1-e^{i(n+1)(\phi-\theta)}\right|^2\left(1-e^{i(\theta-\phi)}\right)}{\left(1-e^{i(\theta-\phi)}\right)^2\left(1-e^{i(\phi-\theta)}\right)^2} d\phi \\
&= B_1^1 + B_1^2.
\end{aligned}$$

By our work done on B_0 we know that $B_1^1 = \pi f(\theta)(n+1)^2 + o(n^2)$. Next, for B_1^2,

$$\begin{aligned}
B_1^2 &= \int_{-\pi}^{\pi} f(\phi) \frac{\left[2 - 2\cos\left((n+1)(\theta-\phi)\right)\right]\left(i\sin(\phi-\theta) + \cos(\phi-\theta) - 1\right)}{\left(2-2\cos(\theta-\phi)\right)^2} d\phi \\
&= -\int_{-\pi}^{\pi} f(\phi) \frac{1 - 1\cos\left((n+1)(\theta-\phi)\right)}{2-2\cos(\theta-\phi)} d\phi \\
&= O(n).
\end{aligned}$$

Thus, it now follows that

$$\overline{B}_1(re^{i\theta}) = \pi\sigma^2 e^{\beta n} f(\theta)(n+1)^2 + o(n^2). \tag{4.19}$$

For A_0 we have the expression

$$\begin{aligned}
A_0 &= \int_{-\pi}^{\pi} f(\phi) \frac{1 - z^{n+1}e^{-i(n+1)\phi}}{1 - ze^{-i\phi}} \cdot \frac{1 - z^{n+1}e^{i(n+1)\phi}}{1 - ze^{i\phi}} d\phi \\
&\sim c_1 \int_{-\theta - \frac{1}{\log(n+1)}}^{-\theta + \frac{1}{\log(n+1)}} f(\phi) \frac{\left(1 - z^{n+1}e^{i(n+1)\phi}\right)\left(1 - \overline{z}e^{-i\phi}\right)}{\left|1 - ze^{i\phi}\right|^2} d\phi \\
&\quad + c_2 \int_{\theta - \frac{1}{\log(n+1)}}^{\theta + \frac{1}{\log(n+1)}} f(\phi) \frac{\left(1 - z^{n+1}e^{-i(n+1)\phi}\right)\left(1 - \overline{z}e^{i\phi}\right)}{\left|1 - ze^{-i\phi}\right|^2} d\phi \\
&\sim c_1 \int_{-\theta - \frac{1}{\log(n+1)}}^{-\theta + \frac{1}{\log(n+1)}} f(\phi) \frac{-\sin\left((n+1)(\theta+\phi)\right)\sin\left(\theta+\phi\right)}{2 - 2\cos\left(\theta+\phi\right)} d\phi \\
&\quad + c_2 \int_{\theta - \frac{1}{\log(n+1)}}^{\theta + \frac{1}{\log(n+1)}} f(\phi) \frac{-\sin\left((n+1)(\theta-\phi)\right)\sin\left(\theta-\phi\right)}{2 - 2\cos\left(\theta-\phi\right)} d\phi \\
&= O(\log n).
\end{aligned}$$

For A_1,

$$\begin{aligned}
A_1 = \int_{-\pi}^{\pi} f(\phi) & \frac{1 - z^{n+1}e^{-i(n+1)\phi}}{1 - ze^{-i\phi}} \\
& \cdot \frac{-(n+1)z^{n+1}e^{i(n+1)\phi}(1 - ze^{i\phi}) + ze^{i\phi}(1 - z^{n+1}e^{i(n+1)\phi})}{(1 - ze^{i\phi})^2} d\phi \\
& \sim c_3(n+1) \int_{\theta - \frac{1}{\log(n+1)}}^{\theta + \frac{1}{\log(n+1)}} f(\phi) \frac{\left(1 - z^{n+1}e^{-i(n+1)\phi}\right)\left(1 - \overline{z}e^{i\phi}\right)}{|1 - ze^{-i\phi}|^2} d\phi \quad + \\
\int_{-\theta - \frac{1}{\log(n+1)}}^{-\theta + \frac{1}{\log(n+1)}} f(\phi) & \left[\frac{c_1(n+1)\left(1 - \overline{z}e^{-i\phi}\right)}{|1 - ze^{i\phi}|^2} + \frac{c_2\left(1 - \overline{z}e^{-i\phi}\right)^2\left(1 - z^{n+1}e^{i(n+1)\phi}\right)}{|1 - ze^{i\phi}|^4} \right] d\phi \\
& \sim \int_{\theta - \frac{1}{\log(n+1)}}^{\theta + \frac{1}{\log(n+1)}} f(\phi) \frac{c_3(n+1)\sin(\theta - \phi)}{2 - 2\cos(\theta - \phi)} d\phi \quad + \\
\int_{-\theta - \frac{1}{\log(n+1)}}^{-\theta + \frac{1}{\log(n+1)}} f(\phi) & \left[\frac{c_1(n+1)\sin(\theta + \phi)}{2 - 2\cos(\theta + \phi)} + \frac{c_2 \sin^2(\theta + \phi)\sin((n+1)(\theta + \phi))}{(2 - 2\cos(\theta + \phi))^2} \right] d\phi \\
& = O\left((n+1)\log(n+1)\right).
\end{aligned}$$

It now follows that

$$\begin{aligned}
F(e^{i\theta}) &= \frac{\overline{B}_1}{D_0} - \frac{\overline{A}_0 A_1}{D_0(B_0 + D_0)} \\
&\sim \frac{\pi f(\theta) n(n+1)}{2\pi f(\theta)(n+1)} \\
&= \frac{n}{2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n+1} \mathrm{E}\nu_n(R(\theta_1, \theta_2)) &= \lim_{n\to\infty} \frac{1}{2\pi(n+1)} \int_{\theta_1}^{\theta_2} \frac{n}{2} d\theta \\
&= \frac{\theta_2 - \theta_1}{4\pi},
\end{aligned}$$

as claimed. □

4.2 An Application to the GSM/EDGE Standard for Mobile Phones

We will now focus on a problem concerning the GSM (Global System for Mobile Communications)/EDGE (Enhanced Data Rates for GSM Evolution) standard for mobile phones. When designing digital receivers for such a system, the properties of the so-called discrete-time overall channel impulse response becomes important. Specifically, the location of the zeros of the z-transform of the discrete-time overall channel impulse response determines the receiver's performance. The randomness inherent in mobile communications results in such a z-transform being a random polynomial. For wireless communications in urban areas it is common for the coefficients of (1) to be mean zero complex Gaussians, with exponentially increasing or decreasing variances (see [26] and the references therein for a more complete discussion). Under these assumptions, Schober and Gerstacker derived explicit results for the location of the zeros when the coefficients are independent. This assumption of independence, however, was made to facilitate the computations. In practice, the authors state that the coefficients will only be approximately uncorrelated.

With that in mind, the goal of this section is to study the behavior of the complex zeros when the coefficients are dependent mean zero complex Gaussians with exponentially increasing or decreasing variances. Using a result from Hughes and Nikeghbali, we will first show that, in the limit, the roots accumulate around a circle in the complex plane, uniformly in the angle, where the radius is determined by the coefficient variances. This behavior holds without any restrictions on the covariance function of the coefficients and corresponds with the behavior observed by Schober and Gerstacker in the independent case. The drawback is that this result applies only to the limiting behavior, and it fails to give any detail as to how fast this occurs or how close to the circle the zeros accumulate. Thus, to get a more detailed analysis we will use the techniques developed by Shepp and Vanderbei. In order for us to apply these techniques when the coefficients are dependent, some concessions must be made. Namely, it will be necessary for us to assume that the covariance function of the coefficients is absolutely summable and that the spectral density does not vanish. Another way to interpret these conditions is that we are requiring fast enough decay for the covariance of the coefficients.

4.2.1 Exponentially Increasing/Decreasing Variances

We will start by giving a result from Hughes and Nikeghbali [15]. Let $P_n(z)$ be of the form given in (1), and let $\nu_n(\Omega)$ be the number of zeros of $P_n(z)$ in the set Ω. Also, for $0 < r < 1$ define the annulus $a(r) = \{z \in \mathbb{C} : 1 - r \leq |z| \leq 1/(1-r)\}$, and for $0 \leq \theta_1 < \theta_2 \leq 2\pi$ let $C(\theta_1, \theta_2)$ be the cone in the complex plane consisting of all points with arguments between θ_1 and θ_2.

Theorem 4.2.1 (Hughes and Nikeghbali)**.** *Assume the coefficients of $P_n(z)$ are complex Gaussians with mean zero and unit variance. Then there exists a deterministic positive sequence (α_n), subject to $0 < \alpha_n \leq n$ for all n and $\alpha_n = o(n)$ as $n \to \infty$, such that*

$$\lim_{n\to\infty} \frac{1}{n}\nu_n\left(a\left(\frac{\alpha_n}{n}\right)\right) = 1, \quad a.s.$$

and

$$\lim_{n\to\infty} \frac{1}{n}\nu_n\left(C(\theta_1, \theta_2)\right) = \frac{\theta_2 - \theta_1}{2\pi}, \quad a.s.$$

In other words, the above theorem tells us that for mean zero complex Gaussian coefficients with unit variance, the zeros will accumulate around the unit circle in the limit, uniformly in the angle. Furthermore, this occurs without any restrictions on the dependence of the coefficients. Now, consider the random polynomial

$$\tilde{P}_n(z) = \sum_{k=0}^{n} \sigma e^{\beta(n-k)/2} Z_k z^k,$$

where the Z_k are mean zero complex Gaussians with unit variance, $\sigma > 0$, and $\beta \in \mathbb{R}$. Thus, the coefficients now have exponentially growing or decaying variances, depending on the value of β. Let z_0 be a root of $P_n(z)$. Then,

$$\begin{aligned}
\tilde{P}_n(e^{\beta/2}z_0) &= \sigma\left(e^{\beta n/2}Z_0 + e^{\beta(n-1)/2}Z_1 e^{\beta/2}z_0 + \ldots + Z_n\left(e^{\beta/2}z_0\right)^n\right) \\
&= \sigma e^{\beta n/2}\left(Z_0 + Z_1 z_0 + \ldots + Z_n z_0^n\right) \\
&= 0,
\end{aligned}$$

and it follows that $e^{\beta/2}z_0$ is a root of $\tilde{P}_n(z)$. Applying Theorem 4.2.1, we can then conclude that the roots of $\tilde{P}_n(z)$ accumulate around a circle of radius $e^{\beta/2}$, uniformly in the angle. Furthermore, the fact that the expected number of zeros of $P_n(z)$ inside the unit circle is equal to the expected number outside implies the same property for $\tilde{P}_n(z)$ and the circle of radius $e^{\beta/2}$.

To summarize, when the coefficients are dependent complex Gaussians with mean zero and exponentially increasing or decreasing variances, we have shown that the zeros will accumulate around the circle of radius $e^{\beta/2}$ in the limit. Additionally, they will do so uniformly in the angle, and the expected number of zeros inside the circle will be equal to the expected number outside. The rest of this section's goal will be to give a more thorough analysis of this behavior. This will be accomplished by imposing some restrictions on the covariance function of the coefficients, which will then allow us to use Shepp and Vanderbei's techniques to give this more detailed discussion.

4.2.2 Derivation of a Formula for Computing Zeros

We will assume from now on that $P_n(z)$ has the form

$$P_n(z) = \sum_{k=0}^{n} (U_k + iV_k) z^k = \sum_{k=0}^{n} Z_k z^k, \tag{4.20}$$

where the coefficients are complex Gaussians with mean zero. In addition, they will have exponentially increasing or decreasing variances; that is,

$$\mathrm{E}\left[Z_k \overline{Z}_k\right] = \sigma_k^2 = \sigma^2 e^{\beta(n-k)}, \tag{4.21}$$

for $0 \le k \le n$, $\sigma > 0$, and $\beta \in \mathbb{R}$. In [26] the coefficients were taken to be independent to simplify the calculations. We will now assume some dependence among the coefficients. Following the explanation given on page 893 in [27], the covariance will be given by

$$\begin{aligned} \mathrm{E}\left[Z_k \overline{Z}_j\right] &= \mathrm{E}\left[(U_k + iV_k)(U_j - iV_j)\right] \\ &= \mathrm{E}\left[U_k U_j\right] + \mathrm{E}\left[V_k V_j\right] \\ &= \frac{\sigma^2 e^{\beta(2n-k-j)/2}}{2}\Gamma(k-j) + \frac{\sigma^2 e^{\beta(2n-k-j)/2}}{2}\Gamma(k-j) \\ &= \sigma^2 e^{\beta(2n-k-j)/2}\Gamma(k-j). \end{aligned} \tag{4.22}$$

Thus,

$$\mathrm{E}\left[U_k U_j\right] = \mathrm{E}\left[V_k V_j\right] = \frac{1}{2}\mathrm{E}\left[Z_k \overline{Z}_j\right]. \tag{4.23}$$

Two additional expressions that we will need are

$$\begin{aligned} B_0(z) &= \mathrm{E}\left[P_n(z)\overline{P_n(z)}\right], \\ B_1(z) &= \mathrm{E}\left[P_n(z)\overline{z}\overline{P_n'(z)}\right]. \end{aligned} \tag{4.24}$$

One main difference from the independent case is that these expressions are not straightforward to compute; they depend on the values of the spectral density. To apply these formulas we will rely heavily on deriving asymptotic values throughout this paper. As before, let $\nu_n(\Omega)$ be the number of zeros of $P_n(z)$ in the set Ω. We are now ready to state our first theorem, which extends Shepp and Vanderbei's result to our particular case.

Theorem 4.2.2. *For any region $\Omega \in \mathbb{C}$ whose boundary intersects the real axis at most finitely many times we have*

$$\mathrm{E}[\nu_n(\Omega)] = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{1}{z} F(z) dz, \tag{4.25}$$

where

$$F(z) = \frac{\overline{B_1(z)}}{B_0(z)}. \tag{4.26}$$

Proof. As noted by the authors in [28], the proof used for real Gaussians can be applied to complex Gaussians, and in which case the computations will simplify. The first part of this proof will carry out these simplified calculations, while the second part will apply the spectral density form of the covariance function to compute the needed expressions.

To start, we can use the argument principle to compute $\nu_n(\Omega)$ (see [6] for a reference). It follows that

$$\nu_n(\Omega) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{P_n'(z)}{P_n(z)} dz.$$

By applying Fubini's Theorem [22] and a result of Hammersley [14] on the distribution of the zeros of a random polynomial with complex Gaussian coefficients, we can take the expectation and move it inside the integral. Thus, we arrive at the formula

$$\begin{aligned}\mathrm{E}[\nu_n(\Omega)] &= \frac{1}{2\pi i}\int_{\partial\Omega} \mathrm{E}\left[\frac{P_n'(z)}{P_n(z)}\right] dz \\ &= \frac{1}{2\pi i}\int_{\partial\Omega} \frac{1}{z}\mathrm{E}\left[\frac{zP_n'(z)}{P_n(z)}\right] dz.\end{aligned}$$

The rest of the proof will be devoted to deriving the formula for $F(z) = \mathrm{E}\left[\frac{zP_n'(z)}{P_n(z)}\right]$ given in (4.26).

Following the procedure in [28], we can decompose $P_n(z)$ and $zP_n'(z)$ into their real and imaginary parts. We have

$$\begin{aligned} P_n(z) &= Y_1 + iY_2, \\ zP_n'(z) &= Y_3 + iY_4, \end{aligned}$$

where

$$(4.27)\qquad \begin{array}{rclrcl} Y_1 &=& \sum_{j=0}^n a_jU_j - b_jV_j, & Y_2 &=& \sum_{j=0}^n b_jU_j + a_jV_j, \\ Y_3 &=& \sum_{j=0}^n c_jU_j - d_jV_j, & Y_4 &=& \sum_{j=0}^n d_jU_j + c_jV_j. \end{array}$$

Also,

$$(4.28)\qquad \begin{array}{rclcl} a_j &=& \mathrm{Re}(z^j) &=& \frac{z^j+\overline{z}^j}{2}, \\ b_j &=& \mathrm{Im}(z^j) &=& \frac{z^j-\overline{z}^j}{2i}, \\ c_j &=& ja_j, && \\ d_j &=& jb_j. && \end{array}$$

Let M denote the covariance matrix of $Y = [\begin{array}{cccc} Y_1 & Y_2 & Y_3 & Y_4 \end{array}]^T$. That is,

$$(4.29)\qquad M = \begin{bmatrix} \mathrm{E}[Y_1^2] & 0 & \mathrm{E}[Y_3Y_1] & \mathrm{E}[Y_4Y_1] \\ 0 & \mathrm{E}[Y_2^2] & \mathrm{E}[Y_3Y_2] & \mathrm{E}[Y_4Y_2] \\ \mathrm{E}[Y_3Y_1] & \mathrm{E}[Y_3Y_2] & \mathrm{E}[Y_3^2] & 0 \\ \mathrm{E}[Y_4Y_1] & \mathrm{E}[Y_4Y_2] & 0 & \mathrm{E}[Y_4^2] \end{bmatrix}.$$

Let L be the Cholesky factor for M, where

$$(4.30)\qquad L = \begin{bmatrix} l_{11} & 0 & 0 & 0 \\ 0 & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{bmatrix}.$$

We then have the decomposition

$$\mathrm{E}\left[YY^T\right] = M = LL^T = \mathrm{E}\left[LWW^TL^T\right],$$

where $W = [\ W_1 \quad W_2 \quad W_3 \quad W_4\ ]^T$ is a vector of four independent standard normal random variables. In addition to L being lower triangular, notice that the above series of equalities also implies that $Y =_d LW$. That is, Y and LW are equal in distribution. Using these results, we have

$$\begin{aligned}\frac{zP_n'(z)}{P_n(z)} &= \frac{Y_3 + iY_4}{Y_1 + iY_2}\\ &=_d \frac{(l_{31} + il_{41})W_1 + (l_{32} + il_{42})W_2 + (l_{33} + il_{43})W_3 + il_{44}W_4}{l_{11}W_1 + il_{22}W_2}.\end{aligned}$$

From here, after a series of manipulations and calculations (the details of which are in [28]), we arrive at the following formula:

$$\begin{aligned}F(z) &= \mathrm{E}\left[\frac{zP_n'(z)}{P_n(z)}\right]\\ &= \frac{l_{32} - l_{41} + i(l_{31} + l_{42})}{i(l_{11} + l_{22})}.\end{aligned} \tag{4.31}$$

Now, from (4.23) and (4.27) notice that

$$\begin{aligned}\mathrm{E}\left[Y_1^2\right] &= \sum_{k=0}^{n}\sum_{j=0}^{n}(a_k a_j \mathrm{E}\left[U_k U_j\right] + b_k b_j \mathrm{E}\left[V_k V_j\right])\\ &= \sum_{k=0}^{n}\sum_{j=0}^{n}(a_k a_j \mathrm{E}\left[V_k V_j\right] + b_k b_j \mathrm{E}\left[U_k U_j\right])\\ &= \mathrm{E}\left[Y_2^2\right],\\ \mathrm{E}\left[Y_3 Y_1\right] &= \sum_{k=0}^{n}\sum_{j=0}^{n}(c_k a_j \mathrm{E}\left[U_k U_j\right] + d_k b_j \mathrm{E}\left[V_k V_j\right])\\ &= \sum_{k=0}^{n}\sum_{j=0}^{n}(c_k a_j \mathrm{E}\left[V_k V_j\right] + d_k b_j \mathrm{E}\left[U_k U_j\right])\\ &= \mathrm{E}\left[Y_4 Y_2\right],\end{aligned}$$

and

$$\mathrm{E}\left[Y_3 Y_2\right] = \sum_{k=0}^{n}\sum_{j=0}^{n}(b_k c_j \mathrm{E}\left[U_k U_j\right] - a_k d_j \mathrm{E}\left[V_k V_j\right])$$

$$= -\sum_{k=0}^{n}\sum_{j=0}^{n}(a_k d_j \mathrm{E}\left[U_k U_j\right] - b_k c_j \mathrm{E}\left[V_k V_j\right])$$

$$= -\mathrm{E}\left[Y_4 Y_1\right].$$

Thus, using (4.29) and (4.30), we can solve for the coefficients of L to get

$$\begin{aligned} l_{11} &= \frac{\mathrm{E}[Y_1^2]}{\sqrt{\mathrm{E}[Y_1^2]}}, & l_{22} &= \frac{\mathrm{E}[Y_1^2]}{\sqrt{\mathrm{E}[Y_1^2]}}, \\ l_{31} &= \frac{\mathrm{E}[Y_3 Y_1]}{\sqrt{\mathrm{E}[Y_1^2]}}, & l_{32} &= \frac{\mathrm{E}[Y_3 Y_2]}{\sqrt{\mathrm{E}[Y_1^2]}}, \\ l_{41} &= \frac{-\mathrm{E}[Y_3 Y_2]}{\sqrt{\mathrm{E}[Y_1^2]}}, & l_{42} &= \frac{\mathrm{E}[Y_3 Y_1]}{\sqrt{\mathrm{E}[Y_1^2]}}. \end{aligned}$$

Plugging these into (4.31), it follows that

$$F(z) = \frac{\mathrm{E}[Y_3 Y_2] + i\mathrm{E}[Y_3 Y_1]}{i\mathrm{E}[Y_1^2]}. \tag{4.32}$$

We will next derive expressions for $\mathrm{E}[Y_1^2]$, $\mathrm{E}[Y_3 Y_1]$, and $\mathrm{E}[Y_3 Y_2]$. By (1.2), (4.24), (4.21), and (4.22), we have formulas for the following covariances:

$$\begin{aligned} B_0(z) &= \sum_{k=0}^{n}\sum_{j=0}^{n} z^k \overline{z}^j \mathrm{E}\left[Z_k \overline{Z}_j\right] \\ &= \sum_{k=0}^{n}\sum_{j=0}^{n} z^k \overline{z}^j \sigma^2 e^{\beta(2n-k-j)/2}\Gamma(k-j) \\ &= \sigma^2 e^{\beta n}\sum_{k=0}^{n}\sum_{j=0}^{n}\int_{-\pi}^{\pi} f(\phi) e^{-i(k-j)\phi} z^k e^{-\beta k/2}\overline{z}^j e^{-\beta j/2} d\phi \\ &= \sigma^2 e^{\beta n}\int_{-\pi}^{\pi} f(\phi)\sum_{k=0}^{n}\sum_{j=0}^{n}\left(e^{-i\phi} z e^{-\beta/2}\right)^k \left(e^{i\phi}\overline{z} e^{-\beta/2}\right)^j d\phi \\ &= \sigma^2 e^{\beta n}\int_{-\pi}^{\pi} f(\phi)\frac{1-(z e^{-\beta/2})^{n+1} e^{-i(n+1)\phi}}{1 - z e^{-\beta/2} e^{-i\phi}} \\ &\quad \cdot \frac{1-(\overline{z} e^{-\beta/2})^{n+1} e^{i(n+1)\phi}}{1-\overline{z} e^{-\beta/2} e^{i\phi}} d\phi, \end{aligned} \tag{4.33}$$

$$\begin{aligned} B_1(z) &= \sum_{k=0}^{n}\sum_{j=0}^{n} j z^k \overline{z}^j \mathrm{E}\left[Z_k \overline{Z}_j\right] \\ &= \sum_{k=0}^{n}\sum_{j=0}^{n} j z^k \overline{z}^j \sigma^2 e^{\beta(2n-k-j)/2}\Gamma(k-j) \\ &= \sigma^2 e^{\beta n}\sum_{k=0}^{n}\sum_{j=0}^{n}\int_{-\pi}^{\pi} f(\phi) e^{-i(k-j)\phi} z^k e^{-\beta k/2} j\overline{z}^j e^{-\beta j/2} d\phi \\ &= \sigma^2 e^{\beta n}\int_{-\pi}^{\pi} f(\phi)\sum_{k=0}^{n}\sum_{j=0}^{n}\left(e^{-i\phi} z e^{-\beta/2}\right)^k j\left(e^{i\phi}\overline{z} e^{-\beta/2}\right)^j d\phi \end{aligned} \tag{4.34}$$

$$= \sigma^2 e^{\beta n} \int_{-\pi}^{\pi} f(\phi) \frac{1-(ze^{-\beta/2})^{n+1}e^{-i(n+1)\phi}}{(1-ze^{-\beta/2}e^{-i\phi})(1-\overline{z}e^{-\beta/2}e^{i\phi})^2}$$
$$\cdot \left[-(n+1)(\overline{z}e^{-\beta/2})^{n+1}e^{i(n+1)\phi}(1-\overline{z}e^{-\beta/2}e^{i\phi})\right.$$
$$\left.+\overline{z}e^{-\beta/2}e^{i\phi}(1-(\overline{z}e^{-\beta/2})^{n+1}e^{i(n+1)\phi})\right] d\phi.$$

Using (4.27) and (4.28), we then have

$$\begin{aligned} \mathrm{E}\left[Y_1^2\right] &= \sum_{k=0}^{n}\sum_{j=0}^{n}\left(a_k a_j \mathrm{E}[U_k U_j] + b_k b_j \mathrm{E}[V_k V_j]\right) \\ &= \sum_{k=0}^{n}\sum_{j=0}^{n}\frac{1}{2}\left(z^k\overline{z}^j + \overline{z}^k z^j\right)\frac{1}{2}\mathrm{E}\left[Z_k\overline{Z}_j\right] \\ &= \frac{1}{4}\sum_{k=0}^{n}\sum_{j=0}^{n}\left(z^k\overline{z}^j\mathrm{E}\left[Z_k\overline{Z}_j\right] + \overline{z}^k z^j\mathrm{E}\left[\overline{Z}_k Z_j\right]\right) \\ &= \frac{1}{2}B_0(z), \end{aligned} \tag{4.35}$$

since $\mathrm{E}[Z_k\overline{Z}_j] = \mathrm{E}[\overline{Z}_k Z_j]$. Similarly,

$$\begin{aligned} \mathrm{E}\left[Y_3 Y_1\right] &= \sum_{k=0}^{n}\sum_{j=0}^{n}\left(j a_k a_j \mathrm{E}[U_k U_j] + j b_k b_j \mathrm{E}[V_k V_j]\right) \\ &= \sum_{k=0}^{n}\sum_{j=0}^{n}\frac{1}{2}j\left(z^k\overline{z}^j + \overline{z}^k z^j\right)\frac{1}{2}\mathrm{E}\left[Z_k\overline{Z}_j\right] \\ &= \frac{1}{4}\sum_{k=0}^{n}\sum_{j=0}^{n}\left(j z^k\overline{z}^j\mathrm{E}\left[Z_k\overline{Z}_j\right] + j\overline{z}^k z^j\mathrm{E}\left[\overline{Z}_k Z_j\right]\right) \\ &= \frac{1}{4}\left[B_1(z) + \overline{B_1(z)}\right], \end{aligned} \tag{4.36}$$

and

$$\begin{aligned} \mathrm{E}\left[Y_3 Y_2\right] &= \sum_{k=0}^{n}\sum_{j=0}^{n}\left(j b_k a_j \mathrm{E}\left[U_k U_j\right] - j a_k b_j \mathrm{E}\left[V_k V_j\right]\right) \\ &= \sum_{k=0}^{n}\sum_{j=0}^{n}\frac{1}{2i}j\left(z^k\overline{z}^j - \overline{z}^k z^j\right)\frac{1}{2}\mathrm{E}\left[Z_k\overline{Z}_j\right] \\ &= \frac{1}{4i}\sum_{k=0}^{n}\sum_{j=0}^{n}\left(j z^k\overline{z}^j\mathrm{E}\left[Z_k\overline{Z}_j\right] - j\overline{z}^k z^j\mathrm{E}\left[\overline{Z}_k Z_j\right]\right) \\ &= \frac{-i}{4}\left[B_1(z) - \overline{B_1(z)}\right]. \end{aligned} \tag{4.37}$$

Plugging (4.35), (4.36), and (4.37) into (4.32) then gives

$$F(z) = \frac{\overline{B_1(z)}}{B_0(z)},$$

as claimed. $\square$

4.2.3 Detailed Results on the Distribution of Zeros

Once we have verified Shepp and Vanderbei's formula for the expected number of zeros when some dependence is assumed among the coefficients, we can discuss some applications. We will proceed as they did, proving a couple of results which illustrate the behavior of the complex zeros. While we are expecting similar behavior as in the independent case, the extra assumption of dependence will force us to rely on the spectral density form of the covariance function, along with several asymptotic results, to show this. We will prove two theorems that give a more detailed description of the accumulation of zeros around the circle of radius $e^{\beta/2}$.

Theorem 4.2.3. *Let $D(r)$ be the disk of radius r centered at 0. For any $s \geq 0$ we have*

$$\begin{aligned} \mathrm{E}\left[\nu_n\left(D\left(e^{\beta/2-s/2(n+1)}\right)\right)\right] &\sim \frac{-(n+1)e^{-s}}{1-e^{-s}} + \frac{e^{-s/(n+1)}}{1-e^{-s/(n+1)}} \\ &\sim (n+1)\frac{1-e^{-s}(1+s)}{s(1-e^{-s})}, \end{aligned}$$

as $n \to \infty$. Note that the first line is an equality in the independent case. Letting $s \to 0$, it follows that

$$\sim (n+1)\left(\frac{1}{2} - \frac{s}{3}\right).$$

Proof. From (4.25) we have

$$\begin{aligned} \mathrm{E}[\nu_n\left(D(r)\right)] &= \frac{1}{2\pi i}\int_{\partial D(r)} \frac{1}{z}F(z)dz \\ &= \frac{1}{2\pi}\int_0^{2\pi} F(re^{i\theta})d\theta, \end{aligned} \tag{4.38}$$

where

$$r = e^{\beta/2-s/2(n+1)}, \quad s \geq 0, \quad z = re^{i\theta},$$

and F is as in (4.26). We will need to determine the asymptotic behavior of $\overline{B_1(z)}$ and $B_0(z)$. Note that we can assume θ is bounded some small distance away from $-\pi$ and π. Otherwise, using the fact that

$$\Gamma(k) = \int_{-\pi}^{\pi} e^{-ik\phi} f(\phi) d\phi = \int_{0}^{2\pi} e^{-ik\phi} f(\phi) d\phi = \int_{-2\pi}^{0} e^{-ik\phi} f(\phi) d\phi$$

for any k, the following results will hold with only minor changes to the arguments used.

Starting first with $B_0(z)$ we have

$$\begin{aligned}
B_0(z) &= \sigma^2 e^{\beta n} \int_{-\pi}^{\pi} f(\phi) \frac{1 - e^{-s/2} e^{i(n+1)(\theta-\phi)}}{1 - e^{-s/2(n+1)} e^{i(\theta-\phi)}} \cdot \frac{1 - e^{-s/2} e^{i(n+1)(\phi-\theta)}}{1 - e^{-s/2(n+1)} e^{i(\phi-\theta)}} d\phi \\
&= \sigma^2 e^{\beta n} \int_{\theta-(n+1)^{-\frac{1}{4}}}^{\theta+(n+1)^{-\frac{1}{4}}} f(\phi) \frac{1 - 2e^{-s/2} \cos\left[(n+1)(\theta-\phi)\right] + e^{-s}}{1 - 2e^{-s/2(n+1)} \cos(\theta-\phi) + e^{-s/(n+1)}} d\phi \\
&\quad + \sigma^2 e^{\beta n} \int_{\theta+(n+1)^{-\frac{1}{4}}}^{\pi} f(\phi) \frac{1 - 2e^{-s/2} \cos\left[(n+1)(\theta-\phi)\right] + e^{-s}}{1 - 2e^{-s/2(n+1)} \cos(\theta-\phi) + e^{-s/(n+1)}} d\phi \\
&\quad + \sigma^2 e^{\beta n} \int_{-\pi}^{\theta-(n+1)^{-\frac{1}{4}}} f(\phi) \frac{1 - 2e^{-s/2} \cos\left[(n+1)(\theta-\phi)\right] + e^{-s}}{1 - 2e^{-s/2(n+1)} \cos(\theta-\phi) + e^{-s/(n+1)}} d\phi \\
&= B_0^1 + B_0^2 + B_0^3.
\end{aligned}$$

For B_0^1 we have,

$$\begin{aligned}
B_0^1 &\sim 2\sigma^2 e^{\beta n} \int_{\theta}^{\theta+(n+1)^{-\frac{1}{4}}} c_n f(\phi) \\
&\qquad \cdot \left(\frac{1}{2 - 2(1 - \frac{s}{2(n+1)} + \frac{s^2}{8(n+1)^2})(1 - \frac{(\theta-\phi)^2}{2}) - \frac{s}{n+1} + \frac{s^2}{2(n+1)^2}} \right) d\phi \\
&\sim 2c_n \sigma^2 e^{\beta n} f(\theta) \int_{\theta}^{\theta+(n+1)^{-\frac{1}{4}}} \frac{d\phi}{(\theta-\phi)^2 + \frac{s^2}{4(n+1)^2}} \\
&= c_n \sigma^2 e^{\beta n} f(\theta) \frac{4}{s}(n+1) \arctan\left(\frac{2}{s}(n+1)(\phi-\theta) \right) \Big|_{\theta}^{\theta+(n+1)^{-1/4}} \\
&\sim c_n \sigma^2 e^{\beta n} f(\theta) \frac{2\pi}{s}(n+1).
\end{aligned}$$

We will next show that B_0^2 and B_0^3 are small compared to B_0^1. For B_0^2,

$$B_0^2 \sim \sigma^2 e^{\beta n} \int_{\theta+(n+1)^{-\frac{1}{4}}}^{\pi} \frac{c f(\phi)}{1 - 2e^{-s/2(n+1)} \cos(\theta-\phi) + e^{-s/(n+1)}} d\phi$$

$$
\begin{aligned}
&\leq \sigma^2 e^{\beta n} \int_{\theta+(n+1)^{-\frac{1}{4}}}^{\pi} \frac{cf(\phi)}{1-2e^{-s/2(n+1)}\cos\left(-(n+1)^{-1/4}\right)+e^{-s/(n+1)}} d\phi \\
&\sim \sigma^2 e^{\beta n} \int_{\theta+(n+1)^{-\frac{1}{4}}}^{\pi} \frac{cf(\phi)}{(n+1)^{-1/2}+\frac{s^2}{4(n+1)^2}} d\phi \\
&\sim \sigma^2 e^{\beta n} c(n+1)^{1/2} \\
&= o\left(B_0^1\right).
\end{aligned}
$$

Similarly, we can also show that $B_0^3 = o\left(B_0^1\right)$. It follows that

$$
B_0(z) \sim B_0^1 \sim c_n \sigma^2 e^{\beta n} f(\theta) \frac{2\pi}{s}(n+1).
$$

In the independent case $f(\theta) \equiv \frac{1}{2\pi}$. Setting the quantity above equal to the value of $B_0(z)$ in the independent case, $\sigma^2 e^{\beta n} \frac{1-e^{-s}}{1-e^{-s/(n+1)}}$, allows us to solve for c_n. Thus,

$$
\sigma^2 e^{\beta n}(n+1)\frac{c_n}{s} \sim \sigma^2 e^{\beta n} \frac{1-e^{-s}}{1-e^{-s/(n+1)}} \Rightarrow c_n = \frac{s}{n+1} \cdot \frac{1-e^{-s}}{1-e^{-s/(n+1)}},
$$

and we have now shown that

$$
B_0(z) \sim 2\pi\sigma^2 e^{\beta n} \frac{1-e^{-s}}{1-e^{-s/(n+1)}} f(\theta). \tag{4.39}
$$

Next, for $B_1(z)$ we have

$$
\begin{aligned}
B_1(z) = \sigma^2 e^{\beta n} \int_{-\pi}^{\pi} &\left[\frac{\left((ze^{-\beta/2})^{n+1} e^{-i(n+1)\phi} - 1\right)(n+1)\left(\overline{z}e^{-\beta/2}e^{i\phi}\right)^{n+1}}{\left(1-ze^{-\beta/2}e^{-i\phi}\right)\left(1-\overline{z}e^{-\beta/2}e^{i\phi}\right)} \right. \\
&\left. + \frac{\left|1-\left(ze^{-\beta/2}\right)^{n+1} e^{-i(n+1)\phi}\right|^2 \left(\overline{z}e^{-\beta/2}e^{i\phi} - |z|^2 e^{-\beta}\right)}{\left(1-ze^{-\beta/2}e^{-i\phi}\right)^2\left(1-\overline{z}e^{-\beta/2}e^{i\phi}\right)^2} \right] f(\phi) d\phi \\
= \sigma^2 e^{\beta n} \int_{-\pi}^{\pi} f(\phi) &\left[\frac{-(n+1)\left(e^{-s/2}e^{i(n+1)(\phi-\theta)} - e^{-s}\right)}{\left(1-e^{-s/2(n+1)}e^{i(\theta-\phi)}\right)\left(1-e^{-s/2(n+1)}e^{i(\phi-\theta)}\right)} \right. \\
&\left. + \frac{\left|1-e^{-s/2}e^{i(n+1)(\theta-\phi)}\right|^2 \left(e^{-s/2(n+1)}e^{i(\phi-\theta)} - e^{-s/(n+1)}\right)}{\left(1-e^{-s/2(n+1)}e^{i(\theta-\phi)}\right)^2\left(1-e^{-s/2(n+1)}e^{i(\phi-\theta)}\right)^2} \right] d\phi \\
\sim \sigma^2 e^{\beta n} \int_{-\pi}^{\pi} &f(\phi) \frac{c_n^1 \cdot (n+1)}{\left(1-e^{-s/2(n+1)}e^{i(\theta-\phi)}\right)\left(1-e^{-s/2(n+1)}e^{i(\phi-\theta)}\right)} d\phi \\
+ \sigma^2 e^{\beta n} &\int_{-\pi}^{\pi} \frac{c_n^2 f(\phi)\left(e^{-s/2(n+1)}e^{i(\phi-\theta)} - e^{-s/(n+1)}\right)}{\left(1-e^{-s/2(n+1)}e^{i(\theta-\phi)}\right)^2\left(1-e^{-s/2(n+1)}e^{i(\phi-\theta)}\right)^2} d\phi \\
= B_1^1 + B_1^2&.
\end{aligned}
$$

From our work on B_0 we know that

$$B_1^1 \sim c_n^1 \sigma^2 e^{\beta n}(n+1)^2 f(\theta)\frac{2\pi}{s}.$$

To handle B_1^2 we can apply a procedure similar to the one used on B_0^1 and B_0^2. We then have

$$\begin{aligned}
B_1^2 &\sim \sigma^2 e^{\beta n}\int_{\theta-(n+1)^{-\frac{1}{4}}}^{\theta+(n+1)^{-\frac{1}{4}}} f(\phi)\frac{c_n^2\left(e^{-s/2(n+1)}e^{i(\phi-\theta)}-e^{-s/(n+1)}\right)}{\left(1-2e^{-s/2(n+1)}\cos\left(\theta-\phi\right)+e^{-s/(n+1)}\right)^2}d\phi \\
&\sim 2f(\theta)\sigma^2 e^{\beta n}\int_{\theta}^{\theta+(n+1)^{-\frac{1}{4}}}\frac{c_n^2\left(\frac{s}{2(n+1)}-\frac{(\theta-\phi)^2}{2}\right)}{\left((\theta-\phi)^2+\frac{s^2}{4(n+1)^2}\right)^2}d\phi \\
&= 2f(\theta)\sigma^2 e^{\beta n}\frac{c_n^2(n+1)^2}{s^2}\left[\frac{-(4s(n+1)+s^2)(\theta-\phi)}{s^2+4(n+1)^2(\theta-\phi)^2}\right. \\
&\quad \left.\left. + \left(\frac{s}{2(n+1)}-2\right)\arctan\left(\frac{2}{s}(n+1)(\theta-\phi)\right)\right]\right|_{\theta}^{\theta+(n+1)^{-\frac{1}{4}}} \\
&\sim 2\pi\sigma^2 e^{\beta n} f(\theta)(n+1)^2\frac{c_n^2}{s^2}.
\end{aligned}$$

Using the fact that in the independent case

$$B_1(z) = \sigma^2 e^{\beta n}\frac{-(n+1)e^{-s}(1-e^{-s/(n+1)})+e^{-s/(n+1)}(1-e^{-s})}{(1-e^{-s/(n+1)})^2},$$

we can again solve for the constants using the same procedure as before. Thus,

$$\sigma^2 e^{\beta n}(n+1)^2\frac{c_n^1}{s}+\frac{c_n^2}{s^2} \sim \sigma^2 e^{\beta n}\frac{-(n+1)e^{-s}\left(1-e^{-s/(n+1)}\right)+e^{-s/(n+1)}\left(1-e^{-s}\right)}{\left(1-e^{-s/(n+1)}\right)^2},$$

from which it follows that

$$B_1(z) \sim 2\pi\sigma^2 e^{\beta n}\frac{-(n+1)e^{-s}\left(1-e^{-s/(n+1)}\right)+e^{-s/(n+1)}\left(1-e^{-s}\right)}{\left(1-e^{-s/(n+1)}\right)^2}f(\theta). \tag{4.40}$$

Lastly, since f is real-valued, it is easy to see that

$$B_1(z) \sim \overline{B_1(z)}$$

as well. Thus, plugging (4.39) and (4.40) into (4.38) gives us

$$\begin{aligned}
\mathrm{E}[\nu_n\left(D(r)\right)] &= \frac{1}{2\pi}\int_0^{2\pi} F(re^{i\theta})d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\frac{\overline{B_1(re^{i\theta})}}{B_0(re^{i\theta})}d\theta
\end{aligned}$$

$$
\begin{aligned}
&\sim \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{-(n+1)e^{-s}}{1-e^{-s}}+\frac{e^{-s/(n+1)}}{1-e^{-s/(n+1)}}\right]d\theta\\
&=\frac{-(n+1)e^{-s}}{1-e^{-s}}+\frac{e^{-s/(n+1)}}{1-e^{-s/(n+1)}}\\
&\sim (n+1)\frac{1-e^{-s}(1+s)}{s(1-e^{-s})}.
\end{aligned}
$$

Letting $s \to 0$, we have

$$\sim (n+1)\left(\frac{1}{2}-\frac{s}{3}\right),$$

as claimed. □

Theorem 4.2.4. *Let $r = e^{\beta/2-1/2(k+1)}$. Then,*

$$\lim_{n\to\infty} \mathrm{E}[\nu_n(D(r))] \sim k+1,$$

as $k \to \infty$.

Proof. From (4.25) we have

$$\mathrm{E}[\nu_n(D(r))] = \frac{1}{2\pi}\int_0^{2\pi} F(re^{i\theta})d\theta,$$

where

$$r = e^{\beta/2-1/2(k+1)}, \quad z = re^{i\theta}.$$

We will start by applying the Lebesgue dominated convergence theorem to the components of F, which results in the formula

$$\lim_{n\to\infty} F(z) = \frac{\overline{C(z)}}{B(z)},$$

where

$$
\begin{aligned}
B(z) &= \lim_{n\to\infty} e^{-\beta n}B_0(z) = \sigma^2\int_{-\pi}^{\pi} f(\phi)\frac{1}{1-ze^{-\beta/2}e^{-i\phi}}\cdot\frac{1}{1-\overline{z}e^{-\beta/2}e^{i\phi}}d\phi,\\
C(z) &= \lim_{n\to\infty} e^{-\beta n}B_1(z) = \sigma^2\int_{-\pi}^{\pi} f(\phi)\frac{1}{1-ze^{-\beta/2}e^{-i\phi}}\cdot\frac{\overline{z}e^{-\beta/2}e^{i\phi}}{(1-\overline{z}e^{-\beta/2}e^{i\phi})^2}d\phi.
\end{aligned}
\tag{4.41}
$$

Applying the Lebesgue dominated convergence theorem once more,

$$\lim_{n\to\infty} \mathrm{E}[\nu_n(D(r))] = \frac{1}{2\pi}\int_0^{2\pi} \lim_{n\to\infty} F(re^{i\theta})d\theta. \tag{4.42}$$

We can apply an analysis similar to the one used for $B_0(z)$ and $B_1(z)$ in the proof of Theorem 4.2.3. Then, for $B(z)$ we have

$$\begin{aligned}
B(z) &= \sigma^2 \int_{-\pi}^{\pi} f(\phi) \frac{1}{\left(1 - e^{-1/2(k+1)} e^{i(\theta-\phi)}\right)\left(1 - e^{-1/2(k+1)} e^{i(\phi-\theta)}\right)} d\phi \\
&\sim \sigma^2 \int_{\theta-(k+1)^{-\frac{1}{4}}}^{\theta+(k+1)^{-\frac{1}{4}}} f(\phi) \frac{1}{1 - 2e^{-1/2(k+1)} \cos(\theta - \phi) + e^{-1/(k+1)}} d\phi \\
&\sim \sigma^2 f(\theta) \int_{\theta-(k+1)^{-\frac{1}{4}}}^{\theta+(k+1)^{-\frac{1}{4}}} \frac{1}{(\theta-\phi)^2 + \frac{1}{4(k+1)^2}} d\phi \\
&\sim 2\pi\sigma^2 f(\theta)(k+1).
\end{aligned}$$

Similarly,

$$\begin{aligned}
C(z) &= \sigma^2 \int_{-\pi}^{\pi} f(\phi) \frac{e^{-1/2(k+1)} e^{i(\phi-\theta)} - e^{-1/(k+1)}}{\left(1 - e^{-1/2(k+1)} e^{i(\theta-\phi)}\right)^2 \left(1 - e^{-1/2(k+1)} e^{i(\phi-\theta)}\right)^2} d\phi \\
&\sim \sigma^2 \int_{\theta-(k+1)^{-\frac{1}{4}}}^{\theta+(k+1)^{-\frac{1}{4}}} f(\phi) \frac{e^{-1/2(k+1)} \cos(\phi - \theta) - e^{-1/(k+1)}}{\left(1 - 2e^{-1/2(k+1)} \cos(\theta-\phi) + e^{-1/(k+1)}\right)^2} d\phi \\
&\sim \sigma^2 \int_{\theta-(k+1)^{-\frac{1}{4}}}^{\theta+(k+1)^{-\frac{1}{4}}} \frac{\frac{1}{2(k+1)} - \frac{(\theta-\phi)^2}{2}}{\left((\theta-\phi)^2 + \frac{1}{4(k+1)^2}\right)^2} d\phi \\
&\sim 2\pi\sigma^2 f(\theta)(k+1)^2.
\end{aligned}$$

Plugging into (4.42),

$$\begin{aligned}
\lim_{n\to\infty} \mathrm{E}[\nu_n(D(r))] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \lim_{n\to\infty} F(re^{i\theta}) d\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\overline{C(re^{i\theta})}}{B(re^{i\theta})} d\theta \\
&\sim \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{2\pi\sigma^2 f(\theta)(k+1)^2}{2\pi\sigma^2 f(\theta)(k+1)} d\theta \\
&\sim k+1.
\end{aligned}$$

□

4.2.4 Conclusions

What we first showed is a general result which gives an idea of the limiting behavior of the zeros of a random polynomial that has dependent mean zero complex Gaussian coefficients with exponentially increasing or decreasing variances. By then adding certain restrictions to the covariance function, we were able to derive more accurate results, which in turn give more detailed information on the way in which this occurs. However, even then we were only able to do this by using approximations and asymptotic values. Without having more specific knowledge of the covariance function and the spectral density, we do not see a way to make these results more exact. On the other hand, if one were to know the exact expression of the spectral density it is likely that even more details on the specifics of the behavior could be obtained.

Chapter 5

Random Sums of Orthogonal Polynomials

In this chapter we will give a discussion of the zeros of random sums of orthogonal polynomials, a discussion which is based on the work of Shiffman and Zelditch [29]. Consider a set $\{p_k(z)\}$ of orthogonal polynomials. Let $Z_0, Z_1, \ldots$ be a sequence of i.i.d. complex Gaussians with mean zero and variance one. Then, a random sum of orthogonal polynomials is a random polynomial of the form

$$P_n(z) = \sum_{k=0}^{n} Z_k p_k(z). \tag{5.1}$$

In order to correctly formulate the results of Shiffman and Zelditch, we will need to give a few definitions. To start, let $\mathcal{P}_n$ be the space of polynomials defined on $\mathbb{C}$, with degree less than or equal to n. For Ω a simply connected bounded domain in $\mathbb{C}$ with real analytic boundary (which will henceforth be called a simply connected bounded $\mathcal{C}^\omega$ domain; see [1] for further reference), we can define the inner product on $\mathcal{P}_n$ by

$$\langle f, \overline{g} \rangle_{\partial\Omega,\rho} := \int_{\partial\Omega} f(z)\overline{g(z)}\rho(z)|dz|, \tag{5.2}$$

where ρ is a weight function, $\rho \in \mathcal{C}^\omega(\partial\Omega)$, the space of real analytic functions on a real analytic boundary $\partial\Omega$.

Now, given a compact set $K \in \mathbb{C}$, the equilibrium measure for this set is defined as the unique probability measure that minimizes the energy

$$I(\mu) = -\int_K \int_K \log|z-w| d\mu(z) d\mu(w)$$

(see [17, 31] for further reference). This measure will be denoted as μ_K. If $\{p_k(z)\}$ is an orthonormal basis of $\mathcal{P}_n$ orthogonalized over a domain Ω satisfying certain properties, Shiffman and Zelditch showed that the zeros of $P_n(z)$ will be distribute themselves in the limit according to the equilibrium measure for $\bar{\Omega}$. By a slight abuse of notation, we will let μ_Ω represent this measure. In the case of the closed unit disk, S^1, this is simply Lebesgue measure on the circle, denoted by δ_{S^1}. This statement will be made more precise in what follows.

If we let $\{p_k(z)\}$ be an orthonormal basis of $\mathcal{P}_n$ according to the inner product in (5.2), we can write any arbitrary $P_n \in \mathcal{P}_n$ in the form of (5.1). A Gaussian measure on $\mathcal{P}_n$ will then be defined by the condition that the Z_k's are i.i.d. complex Gaussians with mean zero and unit variance. This measure will be denoted by $\gamma^n_{\Omega,\rho}$. An expectation with respect to $(\mathcal{P}_n, \gamma^n_{\Omega,\rho})$ will be written as $\mathrm{E}^n_{\partial\Omega,\rho}$. Finally, we need to introduce the normalized distribution of zeros for P_n. This is defined as

$$\widetilde{Z}^n_{P_n} := \frac{1}{n} \sum_{P_n(z)=0} \delta_z.$$

In essence, it measures the zeros of P_n. We are now ready to state the main result of Shiffman and Zelditch.

Theorem 5.0.5 (Shiffman and Zelditch)**.** *Suppose that Ω is a simply connected bounded $\mathcal{C}^\omega$ domain and that ρ is a positive $\mathcal{C}^\omega$ density on $\partial\Omega$. Then,*

$$\mathrm{E}^n_{\partial\Omega,\rho}\left[\widetilde{Z}^n_{P_n}\right] = \mu_\Omega + O\left(\frac{1}{n}\right), \tag{5.3}$$

where μ_Ω is the equilibrium measure of $\bar{\Omega}$.

As a further note on notation, in this context $O(f(n))$ corresponds to a distribution $T_n \in \mathcal{D}'(\mathbb{C})$ such that

$$|\langle T_n, \phi\rangle| \leq c_\phi f(n), \qquad \forall \phi \in \mathcal{D}(\mathbb{C}),$$

where c_ϕ does not depend on n.

This result has motivated our investigation of a similar problem, where the random sums of orthogonal polynomials are composed of the "classic" orthogonal polynomials. These would include the Chebyshev, Legendre, and Hermite polynomials. Since the aforementioned polynomials are all orthogonalized on the real line, or some subset thereof, the given theorem of Shiffman and Zelditch would not apply. Thus, in what follows we will lay the groundwork for an investigation into the zeros of such random sums of orthogonal polynomials. We will also present some results pertaining to the specific case of Chebyshev polynomials of the first kind.

5.1 Random Sums of Orthogonal Polynomials on the Real Line

Our discussion here will be closely based on the work of Shiffman and Zelditch in [29], where the necessary changes are made to handle the case when Ω is a subset of the real line, rather than a simply connected bounded $\mathcal{C}^\omega$ domain in $\mathbb{C}$.

To start, we will formulate Proposition 3.3 from [29], which refers only to the specific case of orthonormal polynomials on the closed unit disk.

Proposition 5.1.1 (Shiffman and Zelditch)**.** *Let $\mu = \delta_{S^1}$ denote Haar measure on S^1, and let $\rho \equiv 1$. Then*

$$\mathrm{E}^n_{S^1,\rho}\left[n\widetilde{Z}^n_{P_n}\right] = \frac{i}{2\pi}\left[\frac{1}{\left(|z|^2-1\right)^2} - \frac{(n+1)^2|z|^{2n}}{\left(|z|^{2n+2}-1\right)^2}\right] dz \wedge d\bar{z}.$$

Furthermore, $\mathrm{E}^n_{S^1,\rho}\left[n\widetilde{Z}^n_{P_n}\right] = n\mu + O(1)$*; that is, for all test forms $\phi \in \mathcal{D}(\mathbb{C})$,*

$$\mathrm{E}^n_{S^1,\rho}\left[\sum_{\{z:P_n(z)=0\}} \phi(z)\right] = \frac{n}{2\pi}\int_0^{2\pi} \phi\left(e^{i\theta}\right) d\theta + O(1).$$

In particular, $\mathrm{E}^n_{S^1,\rho}\left[\widetilde{Z}^n_{P_n}\right] \to \mu$ *in $\mathcal{D}'(\mathbb{C})$.*

Once we have the use of this proposition, the idea is to reduce all the other cases back to the unit disk. In order to accomplish this goal, we must introduce some additional notation. Denoting the unit disk as U and letting $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, for a simply connected bounded domain Ω let

$$\Phi : \widehat{\mathbb{C}} \backslash \Omega \longrightarrow \widehat{\mathbb{C}} \backslash U$$

be a conformal mapping for which $\Phi(\infty) = \infty$ and $\Phi'(\infty) \in \mathbb{R}^+$. Letting * denote the pullback, it is a known result that the equilibrium measure for Ω is then given by

$$\mu_\Omega = \Phi^* \delta_{S^1}, \tag{5.4}$$

or equivalently,

$$\int_\Omega \phi(z) d\mu_\Omega(z) = \frac{1}{2\pi} \int_0^{2\pi} \phi \circ \Phi^{-1} \left(e^{i\theta}\right) d\theta.$$

We will now look more closely at our specific sequence of orthonormal polynomials. For the interval $[-1, 1]$, consider the conformal mapping

$$\Phi(z) = z + \left(z^2 - 1\right)^{1/2}, \tag{5.5}$$

which maps $\mathbb{C} \backslash [-1, 1]$ to $\mathbb{C} \backslash U$. Additionally, $\Phi(\infty) = \infty$, $\Phi'(\infty) = 1$, and Φ takes the interval $[-1, 1]$ to the upper half of the boundary of U. Also, let our weight function ρ be given by

$$\rho(z) = (1 - z)^\alpha (1 - z)^\beta,$$

where $\alpha > -1$, $\beta > -1$. The orthogonal polynomials generated by this weight function are called the Jacobi Polynomials. The specific set of polynomials we will consider are the Chebyshev polynomials of the first kind, which arise when $\alpha = \beta = -\frac{1}{2}$. These are given by

$$\widetilde{T}_k(z) = \frac{1}{2} \left(\Phi^k(z) + \Phi^{-k}(z)\right). \tag{5.6}$$

Note that the $\widetilde{T}_k(z)$'s form an orthogonal set, but are not orthonormal. We will define the orthonormal set of Chebyshev polynomials of the first kind by

$$\begin{aligned} T_0(z) &= \frac{1}{\sqrt{\pi}} \widetilde{T}_0(z), \\ T_k(z) &= \sqrt{\frac{2}{\pi}} \widetilde{T}_k(z), \qquad k > 0. \end{aligned} \tag{5.7}$$

We are now ready to state our main result.

Theorem 5.1.1. *Let $Z_1, Z_2, \dots$ be a sequence of independent complex Gaussians, with mean zero and variance one. Consider the random sum of orthogonal polynomials given by*

$$P_n(z) = \sum_{k=0}^{n} Z_k T_k(z),$$

where $T_k(z)$ is the k-th orthonormal Chebyshev polynomial of the first kind defined above. Let ρ be the weight function given by $\rho(z) = (1-z)^{-1/2}(1+z)^{-1/2}$. Then, for $\Omega = [-1, 1]$,

$$\mathrm{E}^n_{\partial\Omega,\rho}\left(\widetilde{Z}^n_{P_n}\right) = \mu_\Omega + O\left(\frac{1}{n}\right).$$

Proof. In [29], the authors' approach was to first prove the desired result for U, then use a conformal mapping to handle a more general domain. We will follow the same procedure here. The main changes that need to be made are in section 3.4 of their work. Keeping the same notation, we will define

$$S_n(z,z) = \sum_{k=0}^{n} |T_k(z)|^2,$$

and

$$S_n^U(z,z) = \sum_{k=0}^{n} |z|^{2k}.$$

Now, noting that the proof given for Proposition 3.1 in [29] holds for the present case, we have the formula

$$\mathrm{E}^n_{\partial\Omega,\rho}\left[n\widetilde{Z}^n_{P_n}\right] = \frac{i}{2\pi}\partial\bar{\partial}\log S_n(z,z). \tag{5.8}$$

Thus, our main goal will be to show that

$$\frac{i}{2\pi}\partial\bar{\partial}\log S_n(z,z) = n\mu_\Omega + O(1),$$

in $\mathcal{D}'(\mathbb{C})$.

Let

$$A_n(z) = \frac{S_n(z,z)}{\Phi^* S_n^U(z,z)},$$

where

$$\Phi^* S_n^U(z,z) = \sum_{k=0}^{n} |\Phi_n(z)|^{2k} = \begin{cases} n+1 & \text{if } z \in [-1,1] \\ \frac{1-|\Phi(z)|^{2(n+1)}}{1-|\Phi(z)|^2} & \text{if } z \in \mathbb{C}/[-1,1]. \end{cases} \tag{5.9}$$

From (5.7), for $k > 0$ we have

$$(5.10)\qquad \begin{aligned} |T_k(z)|^2 &= \left| \frac{1}{\sqrt{2\pi}} \left(\Phi^k(z) + \Phi^{-k}(z) \right) \right|^2 \\ &= \frac{1}{2\pi} \left[\left| \Phi^k(z) \right|^2 + \left| \Phi^{-k}(z) \right|^2 + 2\mathrm{Re} \left(\Phi^k(z) \overline{\Phi^{-k}}(z) \right) \right]. \end{aligned}$$

Now, consider the ellipse in the complex plane with foci at -1 and 1, and semi-axes given by

$$\frac{1}{2}\left(r + r^{-1}\right), \qquad \frac{1}{2}\left(r - r^{-1}\right),$$

where $r > 1$. We will denote such an ellipse by $e(r)$. $\Phi(z)$ takes an ellipse of this form and maps it to the circle of radius r (see Section 1.9 in [30]). Thus, for a point z of the form

$$(5.11)\qquad z = \frac{1}{2}\left(r + r^{-1}\right)\cos\theta + \frac{i}{2}\left(r - r^{-1}\right)\sin\theta,$$

where $\theta \in [0, 2\pi)$, it follows that

$$\Phi(z) = re^{i\theta}.$$

For z of the form given in (5.11), (5.10) becomes

$$(5.12)\qquad |T_k(z)|^2 = \frac{1}{2\pi}\left[r^{2k} + r^{-2k} + 2\cos(2k\theta)\right].$$

Define

$$E(R) := \{z : z \in e(r), 1 < r < R\}.$$

That is, $E(R)$ is all the points inside the ellipse $e(R)$, with the exception of the line $[-1, 1]$. For $A_n(z)$ we then have

$$(5.13)\qquad \begin{aligned} A_n(z) &= \frac{\frac{1}{\pi} + \frac{1}{2\pi}\sum_{k=1}^{n}\left[r^{2k} + r^{-2k} + 2\cos(2k\theta)\right]}{\sum_{k=0}^{n} r^{2k}} \\ &= \frac{\frac{1}{2\pi}\sum_{k=0}^{n}\left[r^{2k} + r^{-2k}\right] + \frac{1}{\pi}\sum_{k=1}^{n}\cos(2k\theta)}{\sum_{k=0}^{n} r^{2k}} \\ &= \frac{1}{2\pi} + \frac{\frac{1}{\pi}\sum_{k=1}^{n}\cos(2k\theta)}{\sum_{k=0}^{n} r^{2k}} + o(1). \end{aligned}$$

We will next need to make use of the following lemma.

Lemma 5.1.1. *Let m be Lebesgue measure on the measure space $(\Omega, \mathcal{F})$, where $\Omega = [0, 2\pi)$ and $\mathcal{F}$ is the σ-algebra of Lebesgue measurable sets. Then,*

$$m\left(\theta : \left|\sum_{k=1}^{n} \cos(2k\theta)\right| \geq \frac{n}{2}\right) \leq \frac{4\pi}{n}.$$

Proof. For $k > 0$ define

$$X_k(\omega) := \cos(2k\omega).$$

Also, for $A \in \mathcal{F}$ let

$$\mathrm{P}(A) = \frac{1}{2\pi} m(A).$$

Then, on the probability triple $(\Omega, \mathcal{F}, \mathrm{P})$, $X_1, X_2, \ldots$ is a sequence of identically distributed random variables, which, while not independent, are uncorrelated. Since $\mathrm{E}[X_k] = 0$ and $\mathrm{E}[X_k^2] = 1/2$, by Lemma 5.1 in [8] it follows that

$$\mathrm{var}\left(\frac{1}{n}\sum_{k=1}^{n} X_k\right) = \sum_{k=1}^{n} \mathrm{var}\left(\frac{X_k}{n}\right) = \frac{1}{2n}.$$

Thus,

$$\mathrm{P}\left(\left|\frac{1}{n}\sum_{k=1}^{n} X_k\right|^2 \geq \frac{1}{4}\right) \leq \frac{2}{n},$$

which proves the claim. □

Consider the fraction

$$\frac{\frac{1}{2\pi}\sum_{k=1}^{n} 2\cos(2k\theta)}{\sum_{k=0}^{n} r^{2k}}.$$

Let m now represent Lebesgue measure on $\mathbb{C}$. For any $z \in E(R)$, let $z \in B$ if

$$\left|\sum_{k=1}^{n} \cos(2k \arg(z))\right| \geq \frac{n}{2}.$$

By Lemma 5.1.1, $m(B) = O\left(\frac{1}{n}\right)$. Also, notice that the inequality

$$A_n(z) \geq \frac{1}{\pi \sum_{k=0}^{n} r^{2k}} = \frac{r^2 - 1}{\pi\left(r^{2(n+1)} - 1\right)}, \tag{5.14}$$

holds in general. Thus, for $\phi \in \mathcal{D}(\mathbb{C})$ we have

$$\begin{aligned} \left|\int_B \partial\bar{\partial}\phi(z) \log A_n(z) dm(z)\right| &\leq \int_B \left|\partial\bar{\partial}\phi(z) \log\left(\frac{r^2 - 1}{\pi\left(r^{2(n+1)} - 1\right)}\right)\right| dm(z) \\ &\leq c\left|\log\left(\frac{r^2 - 1}{\pi\left(r^{2(n+1)} - 1\right)}\right)\right| m(B) \\ &= O(1). \end{aligned} \tag{5.15}$$

Notice next that if $z \in E(R)\backslash B$, then $A_n(z) = O(1)$. Thus,

$$\left| \int_{E(R)\backslash B} \partial\bar{\partial}\phi(z) \log A_n(z) dm(z) \right| = O(1). \tag{5.16}$$

Finally, suppose $z \in \mathbb{C}\backslash\{E(R) \cup [-1,1]\}$. Then,

$$\frac{\frac{1}{2\pi}\sum_{k=1}^{n} 2\cos(2k\theta)}{\sum_{k=0}^{n} r^{2k}} = o(1),$$

which implies that $A_n(z) \sim c$. Thus,

$$\left| \int_{\mathbb{C}\backslash\{E(R)\cup[-1,1]\}} \partial\bar{\partial}\phi(z) \log A_n(z) dm(z) \right| = O(1) \tag{5.17}$$

as well. Combining (5.15), (5.16), and (5.17), we now have

$$\int_{\mathbb{C}} \partial\bar{\partial}\phi(z) \log A_n(z) dm(z) = O(1),$$

which shows that $\partial\bar{\partial} \log A_n = O(1)$ in $\mathcal{D}'(\mathbb{C})$. Now, for the weight function $w \equiv 1$, by Proposition 3.1 in [29],

$$\mathrm{E}^n_{S^1,w}\left[n\widetilde{Z}^n_{P_n}\right] = \frac{i}{2\pi}\partial\bar{\partial}\log S_n^U(z,z).$$

Applying this result, along with (5.4), (5.8), and Proposition 5.1.1, it follows that

$$\begin{aligned} \frac{i}{2\pi}\partial\bar{\partial}\log S_n(z,z) &= \Phi^*\left(\frac{i}{2\pi}\partial\bar{\partial}\log S_n^U(z,z)\right) - \frac{i}{2\pi}\partial\bar{\partial}\log A_n(z) \\ &= \Phi^*\left(n\mu + O(1)\right) + O(1) \\ &= n\mu_\Omega + O(1). \end{aligned} \tag{5.18}$$

□

5.2 Conclusions

The motivation of this chapter was to lay the foundation for some future work in this area. While we have succeeded in showing that, for the Chebyshev polynomials of the first kind, the zeros of P_n converge to the equilibrium distribution, we believe that similar results should hold for the Legendre polynomials, as well as the rest of the Jacobi polynomials. A further problem that is also of interest is to derive similar results when P_n is composed of the Hermite polynomials. Thus, we hope that this extension of Shiffman and Zelditch's work to the Chebyshev polynomials will lead to further results on the aforementioned problems.

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APPENDICES

Appendix A

To give a clearer view of the types of spectral density functions we may be considering, we have included here graphs of the spectral density for various covariance functions. Since the formula for the spectral density is given as an infinite sum (1.3), figures A.1-A.3 are numerical approximations. Figure A.4 is a graph of the equation given in (1.4). All of these figures were done in Matlab.

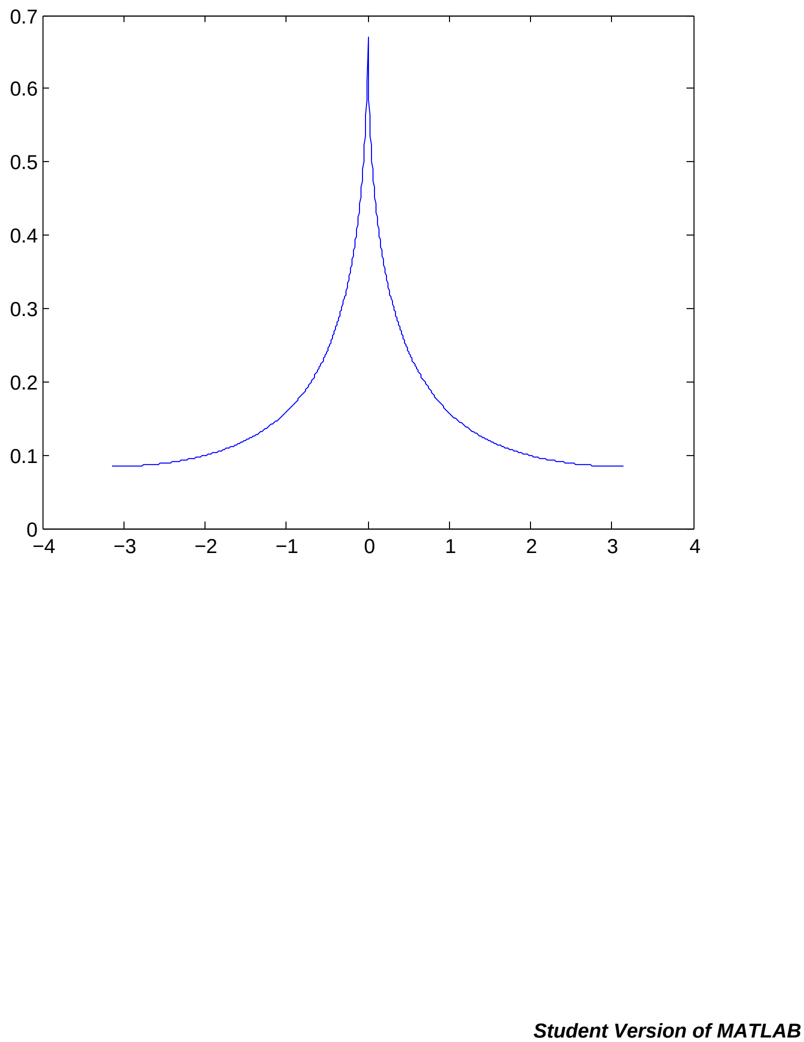

Figure A.1: Spectral density for $\Gamma(k) = \frac{1}{|k|^{3/2}+1}$

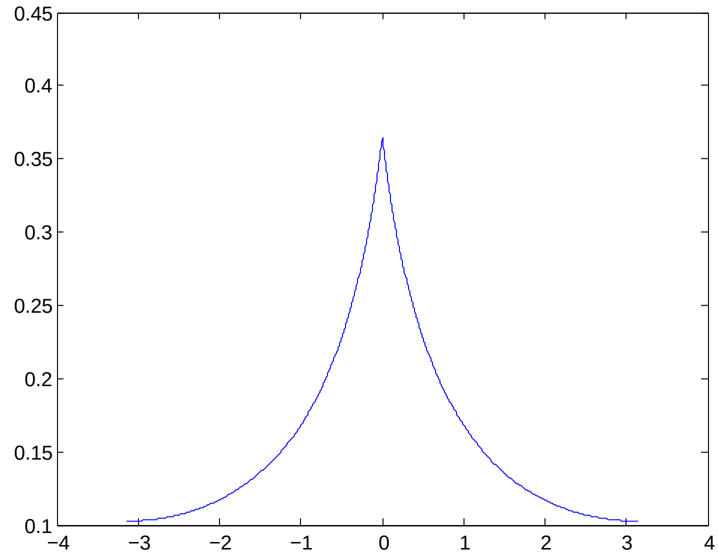

Figure A.2: Spectral density for $\Gamma(k) = \frac{1}{k^2+1}$

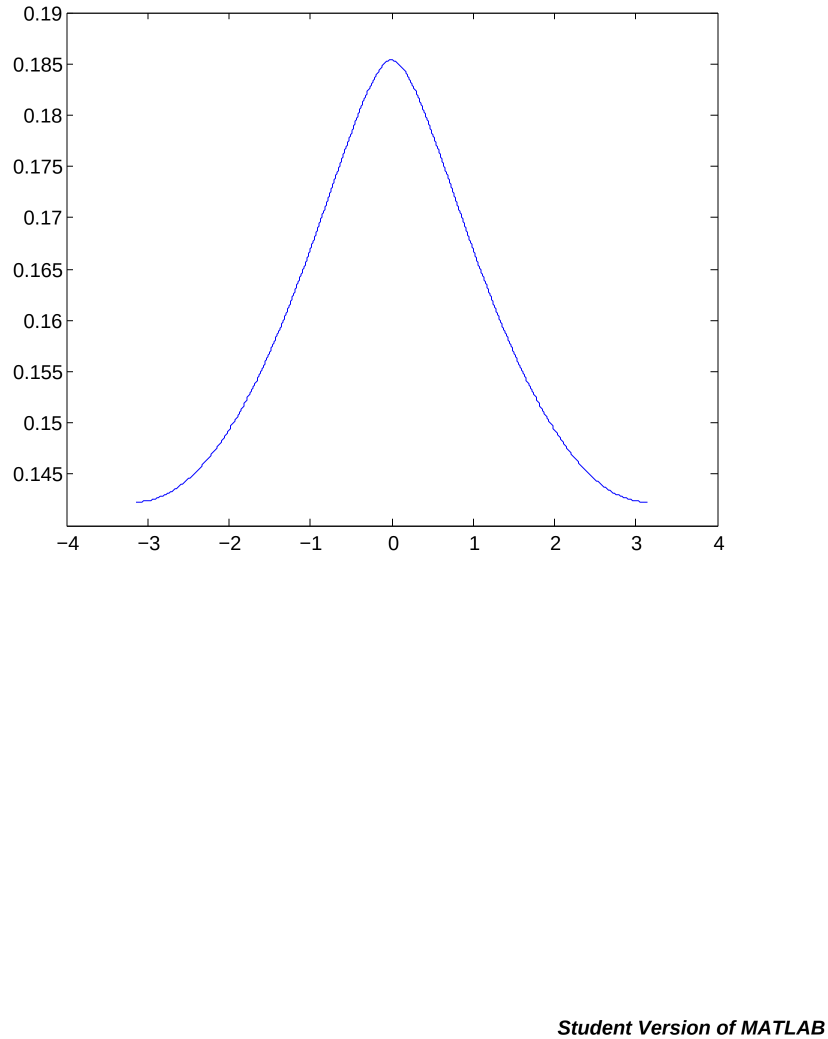

Figure A.3: Spectral density for $\Gamma(k) = \frac{1}{k^4+1}$

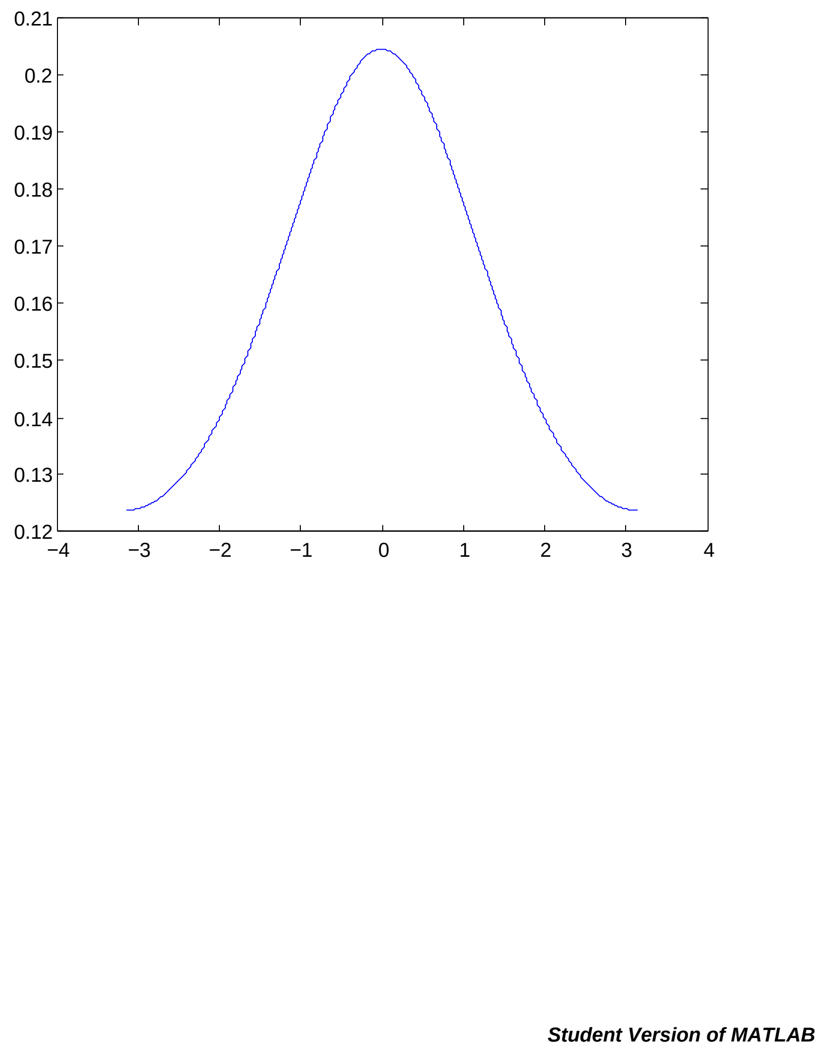

Figure A.4: Spectral density for $\Gamma(k) = \rho^{|k|}$, $\rho = \frac{1}{8}$

Appendix B

The following figures represent numerical simulations of the complex zeros of random polynomials with independent standard normal coefficients. All of the computations were done in Mathematica. In the captions, n represents the degree of the polynomials generated, while m is the number of realizations of these polynomials generated. The polynomials were simulated using normally distributed pseudorandom numbers generated by Mathematica's Random function. The zeros were then numerically approximated using the NSolve function.

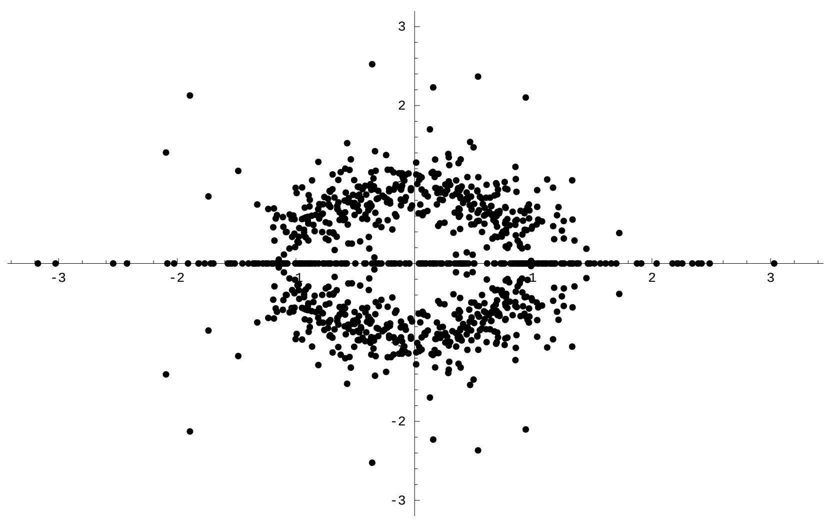

Figure B.1: m=100 n=10

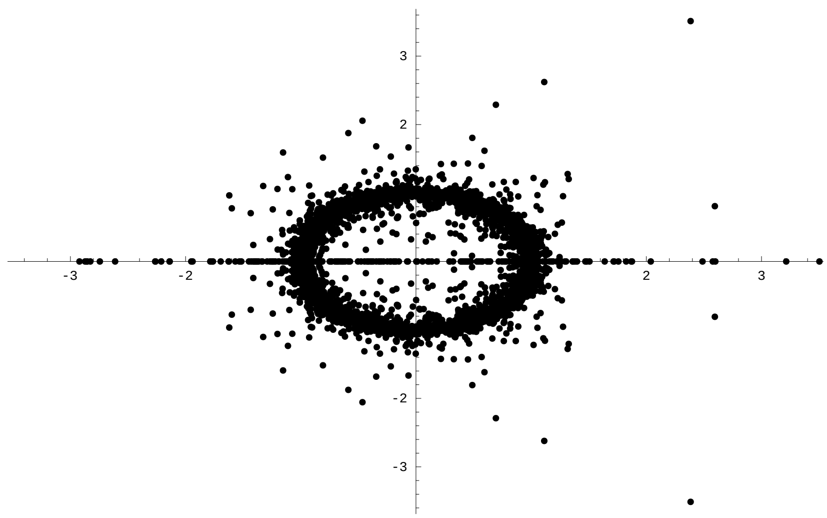

Figure B.2: m=100 n=50

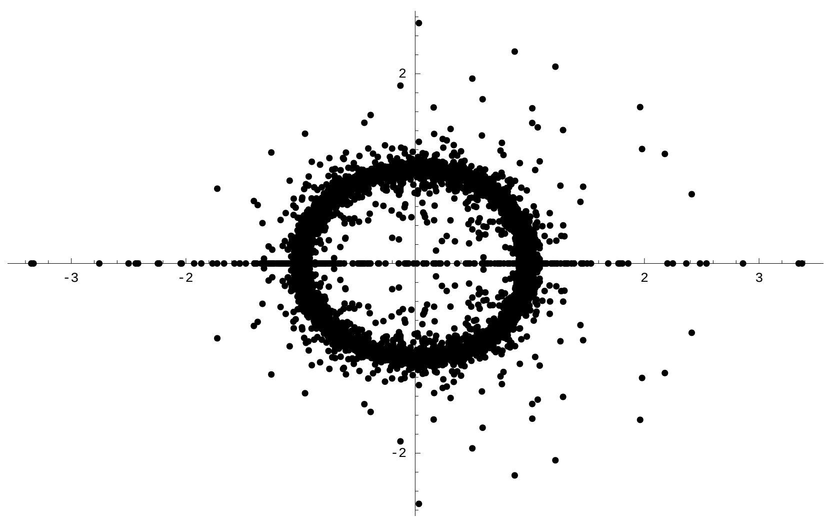

Figure B.3: m=100 n=100